Name KEY

Math 152, Exam #1, Fall 2011

Instructions: Show all work to receive full credit. All proofs must show clear and precise reasoning without assuming steps that aren't shown. If you use a calculator to solve the problem, you must sketch the graph obtained or state the keystrokes used to show work. The use of symbolic manipulator calculators or programs is prohibited on the exam.

1. Integrate. (5 points each)

a. $\int \frac{x^4 - 4x^2 + 1}{x^2} dx$

$\int x^2 - 4 + x^{-2}\, dx = \boxed{\frac{1}{3}x^3 - 4x - \frac{1}{x} + C}$

b. $\int t \cdot \sqrt[3]{t-4}\, dt$

$u = \sqrt[3]{t-4}$

$u^3 = t-4$

$u^3 + 4 = t$

$3u^2\, du = dt$

$(u^3+4)\, u\, 3u^2\, du$

$\int 3u^6 + 12u^3\, du$

$\frac{3}{7}u^7 + 3u^4 + C \longrightarrow \boxed{\frac{3}{7}(t-4)^{7/3} + 3(t-4)^{4/3} + C}$

c. $\int 5\cos x - 2\sec^2 x dx$

$\boxed{5\sin x - 2\tan x + C}$

d. $\int \frac{\arccos(x)}{\sqrt{1-x^2}} dx$

$u = \arccos x$

$-du = \frac{+1}{\sqrt{1-x^2}} dx$

$-\int u\, du = -\frac{u^2}{2} + C$

$-\frac{1}{2}\arccos^2 x + C$

2. Use limit of the Riemann sum to calculate the area under the curve $f(x) = x^2 + 1$ on the interval [-2,3]. [Hint: I suggest using the right-hand rule.] Then **verify** using the Fundamental Theorem of Calculus. (20 points)

$$\Delta x = \frac{3-(-2)}{n} = \frac{5}{n}$$

$$x_i = -2 + \frac{5i}{n}$$

$$f(x_i) = \left(-2+\frac{5i}{n}\right)^2 + 1 = +4 - \frac{20i}{n} + \frac{25i^2}{n^2} + 1$$

$$5 - \frac{20i}{n} + \frac{25i^2}{n^2}$$

$$\sum f(x_i)\Delta x = \sum\left(5 - \frac{20i}{n} + \frac{25i^2}{n^2}\right)\left(\frac{5}{n}\right) =$$

$$\sum \frac{25}{n} - \frac{100}{n^2}\sum i + \frac{125}{n^3}\sum i^2$$

$$= \frac{25\cdot\cancel{n}}{\cancel{n}} - \frac{\overset{50}{\cancel{100}}}{n^{\cancel{2}}}\left(\frac{\cancel{n}(n+1)}{\cancel{2}}\right) + \frac{125}{n^{\cancel{3}}}\,\frac{\cancel{n}\overbrace{(n+1)(2n+1)}^{2n^2+3n+1}}{6}$$

$$\lim_{n\to\infty} 25 - \frac{50\cancel{n}}{\cancel{n}} - \cancel{\frac{50}{n}}^{\,0} + \frac{125\cdot\cancel{2n^2}}{3\,\cancel{6}\cancel{n^2}} + \cancel{\frac{125\cdot\overset{5}{\cancel{3}}n}{6n^2}}^{\,0} + \cancel{\frac{125}{6n^2}}^{\,0}$$

$$= 25 - 50 + \frac{125}{3} = \boxed{\frac{50}{3}}$$

$$\int_{-2}^{3} x^2+1\,dx = \frac{1}{3}x^3 + x\Big|_{-2}^{3} = \frac{1}{3}\cdot 27 + 3 - \frac{1}{3}(-8) - (-2)$$

$$9 + 3 + \frac{8}{3} + 2 = \boxed{50/3}$$

3. Integrate. (5 points each)

a. $\int_0^3 |2x-3|\,dx$

by symmetry

$2\int_{3/2}^{3} 2x-3\,dx = 2\left(x^2-3x\right)\Big|_{3/2}^{3} =$

$2\left[\cancel{9}-\cancel{9}-(3/2)^2+3(3/2)\right] = 2\left[\frac{9}{2}-\frac{9}{4}\right] = 2\cdot\frac{9}{4} =$

$\boxed{\frac{9}{2}}$

b. $\int_1^6 \frac{3}{q}\,dq$ $= 3\ln q\Big|_1^6 = 3\ln 6 - \cancel{3\ln 1} = \boxed{3\ln 6}$

≈ 5.37

c. $\int_0^{\ln 3} x\cdot 3^{x^2}\,dx$

$u = x^2$

$\frac{1}{2}du = \cancel{2}x\,dx$ $\quad \frac{1}{2}\int 3^u\,du$

$\frac{1}{2}\cdot\frac{1}{\ln 3}\,3^{x^2}\Big|_0^{\ln 3} =$

$\frac{1}{2\ln 3}\left[3^{(\ln 3)^2}-3^0\right] =$

$\boxed{\frac{1}{2\ln 3}\left[3^{(\ln 3)^2}-1\right]}$ $\approx$ 1.259

4. Prove by mathematical induction that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. (15 points)

① $n=1$

$\sum_{i=1}^{1} i = 1 \qquad \frac{1(1+1)}{2} = \frac{1\cdot\cancel{2}}{\cancel{2}} = 1$ ✓

checks for $n=1$

② suppose true for n. show true for $n+1$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n+1} i = \frac{(n+1)(n+2)}{2} \quad \text{or} \quad \sum_{i=1}^{n} i + (n+1) = \frac{n(n+1)}{2} + n+1$$

$$\frac{(n+1)(n+2)}{2} \overset{?}{=} \frac{n(n+1)}{2} + n+1$$

$$\frac{n(n+1)}{2} + \frac{2(n+1)}{2}$$

$$\frac{(n+1)[n+2]}{2}$$ ✓

5. Integrate. (5 points each)

a. $\int \frac{e^{1/p}}{p^2}\,dp$

$u = \frac{1}{p} = p^{-1}$

$du = -\frac{1}{p^2}dp$

$-\int e^u\,du = \boxed{-e^{1/p} + C}$

b. $\int (x+1)\sqrt{2-x}\,dx$

$u = \sqrt{2-x}$

$u^2 = 2-x$

$x = 2-u^2 \qquad x+1 = 3-u^2$

$dx = -2u\,du$

$\int -u(3-u^2)2u\,du$

$\int -6u^2 + 2u^4\,du$

$\frac{2}{5}u^5 - 2u^3 + C$

$= \boxed{\frac{2}{5}(2-x)^{5/2} - 2(2-x)^{3/2} + C}$

c. $\int \operatorname{sech}^3 x \tanh x\,dx$

$\operatorname{sech}^2 x \cdot \operatorname{sech} x \tanh x$

$\operatorname{sech} x = u$

$-\operatorname{sech} x \tanh x = du$

$-\int u^2\,du = \frac{-u^3}{3} + C$

$\boxed{-\frac{\operatorname{sech}^3 x}{3} + C}$

d. $\int_{-2}^{2} \frac{dx}{x^2+4x+13}$

$\int \frac{1}{(x+2)^2+9}$

$\frac{1}{3}\arctan\left(\frac{x+2}{3}\right)\Big|_{-2}^{2}$

$\frac{1}{3}\arctan\left(\frac{4}{3}\right) + \cancel{\frac{1}{3}\arctan 0} = \boxed{\frac{1}{3}\arctan\left(\frac{4}{3}\right)}$

$\approx .309$

6. Use the definition of the hyperbolic secant function, in terms of the hyperbolic sine and hyperbolic cosine functions, to prove that $\frac{d}{dz}[\operatorname{sech} z] = -\operatorname{sech} z \tanh z$. (10 points)

$$\operatorname{sech} z = \frac{1}{\cosh z}$$

$$\frac{d}{dz}[\operatorname{sech} z] = \frac{d}{dz}\left[\frac{1}{\cosh z}\right] = \frac{d}{dz}[\cosh z]^{-1} =$$

$$-1[\cosh z]^{-2} \cdot \sinh z = -\frac{1}{\cosh z} \cdot \frac{\sinh z}{\cosh z} =$$

$$\boxed{-\operatorname{sech} z \cdot \tanh z}$$

7. A baseball is thrown upwards from a height of 2 meters with an initial velocity of *10* meters per second. Determine its maximum height. [Hint: find its position function by integrating twice, starting with the acceleration due to gravity = *-9.8* meters/sec^2.] (10 points)

$$\int \overset{a}{-9.8}\, dt =$$

$$-9.8t + C$$

$$\boxed{v = -9.8t + 10}$$

$$\int -9.8t + 10 \, dt$$

$$= -4.9t^2 + 10t + C$$

$$\boxed{s = -4.9t^2 + 10t + 2}$$

$$-9.8t + 10 = 0$$

$$\frac{-9.8t}{-9.8} = \frac{-10}{-9.8}$$

$$t \approx 1.02$$

time at max

$$s(1.02) = -4.9(1.02)^2 + 10(1.02) + 2 =$$

$$\boxed{= 7.10 \text{ meters high}}$$

8. Use the Second Fundamental Theorem of Calculus to evaluate $\frac{dF}{dx}$ for $F(x)=\int_1^{x^3}\sin 2t\,dt$. Then show that you get the same result by integrating first. (10 points)

$\sin 2x^3 \cdot 3x^2$

$\boxed{3x^2\sin(2x^3)}$

vs.

$-\frac{1}{2}\cos 2t\Big|_1^{x^3} = -\frac{1}{2}\cos(2x^3)+\frac{1}{2}\cos(2)$

$\frac{d}{dx}\left[-\frac{1}{2}\cos(2x^3)+\frac{1}{2}\cos(2)\right] =$

$+\frac{1}{\cancel{2}}\sin(2x^3)\cdot\cancel{6}x^2 = \boxed{3x^2\sin(2x^3)}$

9. Find the average value of the function $f(x)=\frac{x^2+4}{x}$, on the interval [1,4]. (5 points)

$\frac{1}{3}\int_1^4 x+\frac{4}{x}\,dx =$

$\frac{1}{3}\left[\frac{1}{2}x^2+4\ln x\right]_1^4 = \frac{1}{3}\left[\frac{1}{2}\cdot 16+4\ln 4-\frac{1}{2}(1)-4\cancel{\ln 1}\right]$

$\frac{1}{3}\left[\underset{\frac{16}{2}}{8}+4\ln 4-\frac{1}{2}\right] =$

$\frac{1}{3}\left[\frac{15}{2}+4\ln 4\right] = \boxed{\frac{5}{2}+\frac{4}{3}\ln 4 \approx 4.348}$

10. Use the Error formulas to calculate the number of partitions needs to calculate the integral $\int_1^3 \frac{1}{\sqrt{x}} dx$ to within 0.000001 for Simpson's Rule. Use: $|E| \le \frac{(b-a)^5}{180n^4}\left[\max\left|f^{IV}(x)\right|\right]$ (10 points)

$f \quad f' \quad f''' \quad f''' \quad f^{IV}$

$x^{-1/2} \to -\frac{1}{2}x^{-3/2} \to \frac{3}{4}x^{-5/2} \to \frac{-15}{8}x^{-7/2} = \frac{105}{16}x^{-9/2}$

$\left|\frac{\cancel{105}}{\cancel{32}\sqrt{x^9}}\right|$ largest at $x=1$

$$\frac{(3-1)^5}{180n^4}\left(\frac{105}{16}\right) \le 10^{-6}$$

32.865 $\le n$

$\frac{\overset{2}{\cancel{32}} \cdot \overset{\overset{7}{\cancel{21}}}{\cancel{105}}}{\underset{\underset{12}{\cancel{36}}}{\cancel{180}} \cdot \cancel{16}}$ $\sqrt[4]{\frac{\cancel{14}}{12}10^6} \le \sqrt[4]{n^4}$

$\Longrightarrow \boxed{n = 34}$ must be even

−2 pts for not rounding
−1 pt for 21

11. $\int_0^{\pi/4} \tan x\,dx$, approximate this integral with the right-hand rule and n=5. (10 points)

$\frac{\pi/4 - 0}{5} = \Delta x = \frac{\pi}{20}$

$0, \frac{\pi}{20}, \frac{\pi}{10}, \frac{3\pi}{20}, \frac{\pi}{5}, \frac{\pi}{4}$

$\left[\underbrace{\cancel{\tan 0}}_{=0} + \tan\frac{\pi}{20} + \tan\frac{\pi}{10} + \tan\frac{3\pi}{20} + \tan\frac{\pi}{5} + \underbrace{\tan\frac{\pi}{4}}_{=1}\right]\frac{\pi}{20}$

$\approx .42715$

12. Integrate. (5 points each) [Hint: the definite integrals may be assisted by properties of even & odd functions.]

a. $\int_{-4}^{4} x^3 + 6x^2 - 2x - 3dx$

$\cancel{\int_{-4}^{4} x^3 - 2x\, dx}_{0} + \int_{-4}^{4} 6x^2 - 3\, dx =$

$2\int_0^4 6x^2 - 3\, dx = 2\left[2x^3 - 3x\right]_0^4 =$

$2[\underbrace{2(64)}_{128} - \underbrace{12}_{-12}] = 2(116) = \boxed{232}$

b. $\int \frac{x(x-2)}{(x-1)^3} dx$ $\quad u = (x-1) \quad du = dx$

$u+1 = x$

$u-1 = x-2$

$\int \frac{(u+1)(u-1)}{u^3} du = \frac{u^2-1}{u^3} du = \int \frac{1}{u} - u^{-3}\, du =$

$\ln u + \frac{1}{2}u^{-2} + C \quad \boxed{\ln|x-1| + \frac{1}{2(x-1)^2} + C}$

c. $\int \frac{\log_4 \varphi}{\varphi} d\varphi$ $\quad u = \log_4 \varphi \quad \log_4 \varphi = \frac{\ln \varphi}{\ln 4}$

$du = \frac{1}{(\ln 4)\varphi} d\varphi \quad \ln 4\, du = \frac{1}{\varphi} d\varphi$

$\int (\ln 4)\, u\, du$

$\ln 4\left(\frac{u^2}{2}\right) + C = \boxed{(\ln 4)[\log_4 \varphi]^2 + C}$

d. $\int_{-\pi}^{\pi} \tan u du$ $\quad$ odd $\quad \boxed{= 0}$