Name KEY

Math 152, Exam #2, Fall 2011

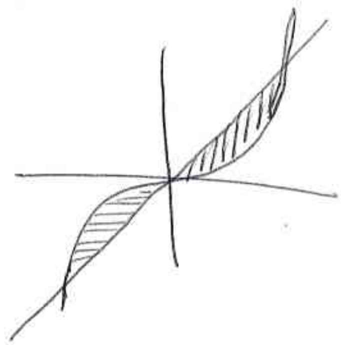
Instructions: Show all work to receive full credit. All proofs must show clear and precise reasoning without assuming steps that aren't shown. If you use a calculator to solve the problem, you must sketch the graph obtained or state the keystrokes used to show work. The use of symbolic manipulator calculators or programs is prohibited on the exam.

1. Find the area bounded by the curves $y = x^3$ and $y = x$. Sketch the graph of the region. (10 points)

$$\int_0^1 x - x^3\,dx$$

$$\frac{1}{2}x^2 - \frac{1}{4}x^4\Big|_0^1 =$$

$$\frac{1}{2} - \frac{1}{4} = \boxed{\frac{1}{4}}$$ just right side



$$\frac{1}{4} + \frac{1}{4} = \boxed{\frac{1}{2}}$$ total area

$$\int_{-1}^0 x^3 - x\,dx =$$

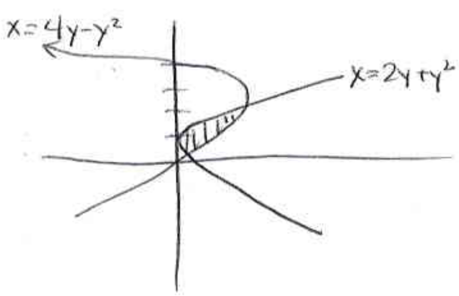
$$\frac{1}{4}x^4 - \frac{1}{2}x^2\Big|_{-1}^0 = -\frac{1}{4} + \frac{1}{2} = \boxed{\frac{1}{4}}$$ left side only

$x = x^3$

$x^3 - x = 0 \quad x(x^2-1) = 0$

$x = 0 \quad (x^2-1) = 0 \quad x = \pm 1$

2. Find the area bounded by the graphs of $x = 4y - y^2$ and $x = 2y + y^2$. Sketch the graph of the region. (10 points)



$$4y - y^2 = 2y + y^2$$
$$-4y + y^2 \quad -4y + y^2$$
$$0 = -2y + 2y^2$$
$$0 = 2y(-1 + y)$$
$$2y = 0 \quad y - 1 = 0$$
$$y = 0 \quad y = 1$$

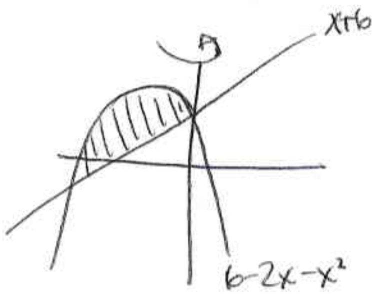
$$\int_0^1 4y - y^2 - (2y + y^2)\,dy = \int_0^1 4y - y^2 - 2y - y^2\,dy =$$

$$\int_0^1 2y - 2y^2\,dy = y^2 - \frac{2}{3}y^3\Big|_0^1 = 1 - \frac{2}{3} = \boxed{\frac{1}{3}}$$

3. Use the **shell method** to find the volume of the solid of revolution generated by revolving the curves $y=6-2x-x^2$ and $y=x+6$ around the y-axis. Sketch the graph of the region. (10 points)

$$6-2x-x^2 = x+6$$
$$-6-x \qquad -x-6$$
$$+3x+x^2=0$$
$$x(x+3)=0$$
$$x=0 \quad x=-3$$

$$2\pi\int -x\left[(6-2x-x^2)-(x+6)\right]dx$$



$$= 2\pi\int -x(\cancel{6}-2x-x^2-x\cancel{-6})\,dx =$$

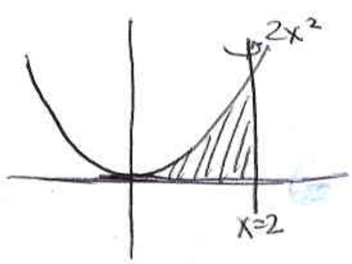
$$2\pi\int -x(-3x-x^2)\,dx =$$

$$2\pi\int 3x^2+x^3\,dx =$$

$$2\pi\left[x^3+\tfrac{1}{4}x^4\right]_0^3 = 2\pi\left[0-\left(-27+\tfrac{81}{4}\right)\right]=$$

$$\boxed{\tfrac{27}{2}\pi \approx 42.41}$$

4. Use the **disk/washer method** to find the volume of the solid generated by revolving region bounded by the curves $y=2x^2, y=0, x=2$ around the line x=2. Sketch the graph. **Set up the integral, but do not integrate.** (10 points)



$$\sqrt{\tfrac{y}{2}}=\sqrt{x^2} \qquad y=8$$

$$x=\frac{\sqrt{y}}{\sqrt{2}}$$

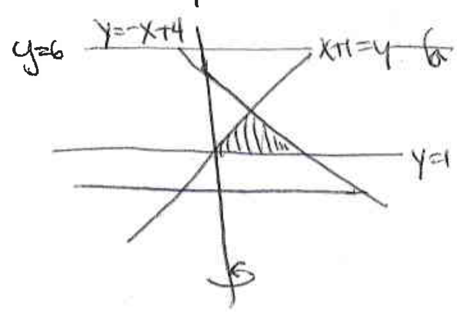
$$\pi\int_0^8 \left(\frac{\sqrt{y}}{\sqrt{2}}-2\right)^2-(2-2)^2\,dy$$

$$=\pi\int_0^8\left(\sqrt{\tfrac{y}{2}}-2\right)^2 dy$$

5. Consider the region bounded by the graphs of $x-y=-1, x+y=4$ and $y=1$. For each case below, **explain which method** (shell or disk/washer) would be best to use to find the volume of revolution **and why**. You do not have to set up any integrals, but it would probably help to sketch the region. Assume that you are equally skilled at both methods: why would a mathematician choose one method or the other? (5 points each)

a. Revolved around the y-axis.

$x+1=y \quad y=-x+4$



washer method
Clear inner & outer radius for the whole region
using shell would need to be split into 2 parts. linear equations are easy to solve for here.

b. Revolved around the line y=6.

Shell method
using washer would entail splitting into 2 regions
Shell can be done w/ one top & one bottom
(or here, one right & one left) function.

6. Calculate the length of arc on the curve $f(x)=x^{3/2}$ on the interval [1,3]. (10 points)

$$f'(x) = \frac{3}{2}x^{1/2}$$

$$S = \int_1^3 \sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}\,dx$$

$$= \int_1^3 \sqrt{1+\frac{9}{4}x}\,dx$$

$u = 1+\frac{9}{4}x$

$du = \frac{9}{4}dx$

$\frac{4}{9}du = dx$

$x=3$

$u = 1+\frac{9}{4}(3) = 31/4$

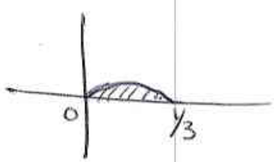
$x=1$

$u = 1+\frac{9}{4}(1) = 13/4$

$$\frac{4}{9}\int_{13/4}^{31/4} u^{1/2}\,du$$

$$\frac{4}{9}\frac{2}{3}u^{3/2}\Big|_{13/4}^{31/4} = \boxed{\frac{8}{27}\left[\left(\frac{31}{4}\right)^{3/2} - \left(\frac{13}{4}\right)^{3/2}\right] \approx 4.6566}$$

7. An ornamental light bulb is designed by resolving the graph of $y = \frac{1}{3}x^{1/2} - x^{3/2}$ on the interval [0,1/3] around the x-axis, where x and y are measured in feet. Find the surface area of the bulb and use the result to approximate the amount of glass needed to make the bulb by multiplying the surface area by the thickness (assume that the glass is 0.015 inches thick). **Be careful about units here!** (20 points)



$$SA = 2\pi \int_0^{1/3} \left(\tfrac{1}{3}x^{1/2} - x^{3/2}\right)\sqrt{1 + \left(\tfrac{1}{6}x^{-1/2} - \tfrac{3}{2}x^{1/2}\right)^2}\, dx$$

$$\approx .116355\ldots$$

$$f'(x) = \tfrac{1}{6}x^{-1/2} - \tfrac{3}{2}x^{1/2}$$

$$V = SA * \text{thickness} = .116355 * .00125 = \boxed{.0107}\ ft^3$$

$$\text{thickness} = \frac{.015\,in}{12\,in/ft} = .00125$$

$18.49\ in^3$

8. A force of 800 newtons stretches a spring 70 centimeters on a mechanical device for driving fence posts. Find the work done in stretching the spring the required 70 centimeters. (10 points)

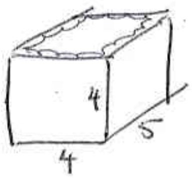
$$\frac{800}{70} = \frac{k\cancel{70}}{\cancel{70}}$$

$$k = \frac{80}{7}$$

$$\int_0^{70} \frac{80}{7}x\, dx =$$

$$\frac{40}{7}x^2 \Big|_0^{70} = \frac{40}{7}(70)^2 = \boxed{28{,}000\ N_{cm}}$$

9. A rectangular tank with a base 4 feet by 5 feet and a height of 4 feet is full of water. The water weighs 62.4 pounds per cubic foot. How much work is done in pumping water out over the top edge in order to empty half of the tank? **Set up the integral, but do not integrate.** (15 points)



$V = 4 \times 5 * dy$
$= 20\,dy$

$F = V * \text{weight} = 20 * 62.4 * dy$
$= 1248\,dy$

$$\int_2^4 (4-y)1248\,dy$$

10. Integrate (5 points each)

a. $\int x\cos(x^2)\,dx$

$u = x^2$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$\frac{1}{2}\int \cos u\,du$

$= \frac{1}{2}\sin u + C$

$\boxed{= \frac{1}{2}\sin(x^2) + C}$

b. $\int \frac{2x}{x-4}dx =$

$$\begin{array}{r} 2 \\ x-4 \overline{)\cancel{2x} + 0} \\ -\cancel{2x} + 8 \\ \hline 8 \end{array}$$

$\int 2 + \frac{8}{x-4}\,dx$

$\boxed{2x + 8\ln|x-4| + C}$

c. $\int \frac{1}{\sqrt{1-4x-x^2}}dx$ $= \int \frac{1}{\sqrt{1-(4+4x+x^2)+4}} =$

$\int \frac{1}{\sqrt{5-(x+2)^2}} dx$

$u = x+2$
$du = dx$
$a^2 = 5$
$a = \sqrt{5}$

$\boxed{= \arcsin\left(\frac{x+2}{\sqrt{5}}\right) + C}$

d. $\int \tan(x)\ln(\cos x)dx$

$u = \ln(\cos x)$
$du = \frac{1}{\cos x} \cdot -\sin x = -\tan x\, dx$
$-du = \tan x\, dx$

$-\int u\, du$

$= -\frac{1}{2}u^2 + C = \boxed{-\frac{1}{2}[\ln(\cos(x))]^2 + C}$

e. $\int x^2 \operatorname{sech}^2(x^3)dx$

$u = x^3$
$du = 3x^2 dx$
$\frac{1}{3}du = x^2 dx$

$= \frac{1}{3}\int \operatorname{sech}^2 u\, du$

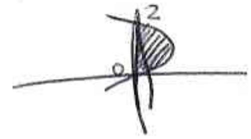
$= \frac{1}{3}\tanh u + C = \boxed{\frac{1}{3}\tanh(x^3) + C}$

f. $\int \frac{x^3-2x^2+1}{x^2}dx$ $= \int \frac{x^3}{x^2} - \frac{2x^2}{x^2} + \frac{1}{x^2}\, dx =$

$\int x - 2 + x^{-2}\, dx$

$\frac{1}{2}x^2 - 2x - x^{-1} + C = \boxed{\frac{1}{2}x^2 - 2x - \frac{1}{x} + C}$

11. Find the centroid of the lamina of uniform density bounded by the graphs $x = 2y - y^2, x = 0$. Write your answer in the form of coordinate points. (25 points).



$$M_y = \frac{1}{2}\rho \int_0^2 (2y-y^2)^2 - 0^2 \, dy$$

$$\frac{1}{2}\rho \int_0^2 4y^2 - 4y^3 + y^4 \, dy =$$

$$\frac{1}{2}\rho \left[\frac{4}{3}y^3 - y^4 + \frac{1}{5}y^5 \right]_0^2 = \frac{1}{2}\rho \left[\frac{4}{3}\cdot 8 - 16 + \frac{1}{5}\cdot 32 \right] = \frac{8}{15}\rho$$

$$M_x = \rho \int_0^2 y(2y-y^2)\, dy = \rho \int_0^2 2y^2 - y^3 \, dy = \rho \left[\frac{2}{3}y^3 - \frac{1}{4}y^4 \right]_0^2 =$$

$$\rho \left[\frac{2}{3}\cdot 8 - \frac{1}{4}\cdot 16 \right] = \frac{4}{3}\rho$$

$$M = \rho \int_0^2 2y - y^2 \, dy = \rho \left[y^2 - \frac{1}{3}y^3 \right]_0^2 = \rho \left[4 - \frac{1}{3}\cdot 8 \right] = \frac{4}{3}\rho$$

$$\bar{x} = \frac{M_y}{M} = \frac{\frac{8}{15}\cancel{\rho}}{\frac{4}{3}\cancel{\rho}} = \frac{\overset{2}{\cancel{8}}}{\underset{5}{\cancel{15}}} \cdot \frac{\cancel{3}}{\cancel{4}} = \frac{2}{5}$$

$$\bar{y} = \frac{M_x}{M} = \frac{\frac{4}{3}\rho}{\frac{4}{3}\rho} = 1$$

$$(\bar{x}, \bar{y}) = \left(\frac{2}{3}, 1\right)$$

center of mass

Check: it makes sense that the y-coordinate is 1 since the graph is symmetric around its axis of symmetry which is $y=1$