Final Exam Study Guide Key

1. $\int \frac{x^3-2x^2+1}{x^2}\,dx = \int \frac{x^3}{x^2} - \frac{2x^2}{x^2} + \frac{1}{x^2}\,dx = \int x - 2 + x^{-2}\,dx = \frac{1}{2}x^2 - 2x - \frac{1}{x} + C$

2. $\int \frac{x+3}{(x^2+6x-5)^2}\,dx$

$u = x^2+6x-5$

$\frac{1}{2}du = \frac{(2x+6)}{2}dx$

$\frac{1}{2}du = (x+3)\,dx$

$\int \frac{\frac{1}{2}du}{u^2} = -\frac{1}{2}\cdot\frac{1}{u} + C$

$= \frac{-1}{2(x^2+6x-5)} + C$

3. $\int \frac{1}{3+25x^2}\,dx$

$a=\sqrt{3}$

$u = 5x$

$du = 5dx \Rightarrow \frac{1}{5}du = dx$

$\frac{1}{5}\int \frac{du}{3+u^2}\,du = \frac{1}{5}\cdot\frac{1}{\sqrt{3}}\arctan\left(\frac{5x}{\sqrt{3}}\right) + C$

4. $\int \frac{x^4+2x^2+x+1}{(x^2+1)^2}\,dx = \int \frac{x^4+2x^2+x+1}{x^4+2x^2+1}\,dx$

$$\begin{array}{r} 1 \\ x^4+2x^2+1 \,\big)\, \cancel{x^4}+\cancel{2x^2}+x+\cancel{1} \\ -\cancel{x^4}-\cancel{2x^2}\quad-\cancel{1} \\ \hline x \end{array}$$

$\int 1 + \frac{x}{(x^2+1)^2}\,dx$

$u = x^2+1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$\frac{1}{2}\int u^{-2}\,du$

$x + -\frac{1}{2}\frac{1}{x^2+1} + C$

5. $\int \frac{x-28}{x^2-x-6}\,dx = \int \frac{A}{x-3} + \frac{B}{x+2}\,dx = \int \frac{-5}{x-3} + \frac{6}{x+2}\,dx$

$(x-3)(x+2)$

$A(x+2) + B(x-3) = x-28$

$x=-2$

$B(-5) = -30 \qquad B=6$

$x=3$

$A(5) = -25 \qquad A=-5$

$-5\ln|x-3| + 6\ln|x+2| + C$

6. $\displaystyle\int \frac{3x^2+1}{(x^2+4)(x-1)}\,dx = \int \frac{Ax+B}{x^2+4} + \frac{C}{x-1}\,dx = \int \frac{\frac{12}{5}x}{x^2+4} + \frac{9/5}{x^2+4} + \frac{4/5}{x-1}\,dx$

$u = x^2+4$
$du = 2x\,dx$
$\frac{1}{2}du = x\,dx$

$(Ax+B)(x-1) + C(x^2+4) = 3x^2+1$

$x=1$ $\quad 5C = 4 \quad C = 4/5$

$x=0$ $\quad -B + \frac{4}{5}(4) = 1 - \frac{16}{5} = -\frac{9}{5} \quad B = 9/5$

$x=2$ $\quad (2A + 9/5)(1) + 4/5(8) = 13$

$2A + 9/5 + \frac{32}{5} = 13 \Rightarrow 10A + 41 = 65 \; (-41) \Rightarrow 10A = 24 \quad A = \frac{24}{10} = \frac{12}{5}$

$$\boxed{\frac{6}{5}\ln|x^2+4| + \frac{9}{10}\arctan\left(\frac{x}{2}\right) + \frac{4}{5}\ln|x-1| + C}$$

7. $\displaystyle\int \cot^4\alpha \csc^2\alpha\, d\alpha$ $\quad u = \cot\alpha$, $du = -\csc^2\alpha\, d\alpha$ $\quad -\int u^4\,du = -\frac{1}{5}u^5 + C = \boxed{-\frac{1}{5}\cot^5\alpha + C}$

8. $\displaystyle\int \frac{\tan^{-1}\left(\frac{x}{2}\right)}{4+x^2}\,dx$ $\quad u = \tan^{-1}\left(\frac{x}{2}\right)$ $\quad du = \frac{1}{1+\left(\frac{x}{2}\right)^2} = \frac{4}{4+x^2}\,dx$

$\frac{1}{4}du = \frac{1}{4+x^2}\,dx$

$\displaystyle\int \frac{1}{4}u\,du = \frac{1}{8}u^2 + C = \frac{1}{8}\left[\arctan\left(\frac{x}{2}\right)\right]^2 + C$

9. $\displaystyle\int \frac{2}{e^{2t}\sqrt{e^{4t}-9}}\,dt$ $\quad$ if $u = e^{2t}$ $\quad du = 2e^{2t}$ $\Rightarrow \displaystyle\int \frac{2e^{2t}}{e^{2t}\cdot e^{2t}\sqrt{e^{4t}-9}}\,dt$

multiply by e^{2t}/e^{2t}

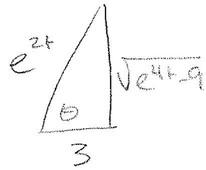
$\displaystyle\int \frac{du}{u^2\sqrt{u^2-9}}$ $\quad u = 3\sec\theta \quad u^2 = 9\sec^2\theta$

$du = 3\sec\theta\tan\theta$

$\sqrt{u^2-9} = 3\tan\theta$

$\displaystyle\int \frac{\cancel{3\sec\theta\tan\theta}\,d\theta}{9\sec^2\theta\,\cancel{3\tan\theta}} = \frac{1}{9}\int \frac{1}{\sec\theta}\,d\theta = \frac{1}{9}\int \cos\theta\,d\theta = \frac{1}{9}\sin\theta + C$

$\frac{e^{2t}}{3} = \frac{u}{3} = \sec\theta$



$$= \boxed{\frac{1}{9}\frac{\sqrt{e^{4t}-9}}{e^{2t}} + C}$$

10. $\displaystyle\int \sin^2\pi t\,dt = \int \frac{1}{2}(1-\cos 2\pi t)\,dt = \boxed{\frac{1}{2}\theta - \frac{1}{4\pi}\sin 2\pi t + C}$

11. $\displaystyle\int \sec^3\theta\,d\theta$ this is done as an example in the textbook so I won't replicate it here.

12. $\displaystyle\int x\ln x\,dx$ $\quad u = \ln x \quad dv = x\,dv$
$du = \frac{1}{x}\,dx \quad v = \frac{1}{2}x^2$

$\frac{1}{2}x^2\ln x - \displaystyle\int \frac{1}{2}x^2\cdot\frac{1}{x}\,dx = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx = \boxed{\frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C}$

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13. $\int e^x \cos x \, dx$ $\quad u = \cos x \quad dv = e^x dx$
$\quad du = -\sin x \, dx \quad v = e^x$

$= e^x \cos x + \int \sin x \, e^x \, dx \;=\; e^x \cos x + \sin x \, e^x - \int e^x \cos x \, dx$
$\quad u = \sin x \quad dv = e^x dx$
$\quad du = \cos x \quad v = e^x$

$\int e^x \cos x \, dx = e^x \cos x + e^x \sin x - \int e^x \cos x \, dx$
$+\int e^x \cos x \, dx \qquad\qquad + \int e^x \cos x \, dx$

$\cancel{2}\int e^x \cos x \, dx = \boxed{\dfrac{e^x \cos x + e^x \sin x}{2} + C}$

14. $\int x\sqrt{4x-1} \, dx$ $\quad$ by parts
$\quad u = x \quad dv = (4x-1)^{1/2}$
$\quad du = dx \quad v = \frac{1}{4}\cdot\frac{2}{3}(4x-1)^{3/2} = \frac{1}{6}(4x-1)^{3/2}$

$\frac{1}{6}x(4x-1)^{3/2} - \frac{1}{6}\int(4x-1)^{3/2}\,dx = \frac{1}{6}x(4x-1)^{3/2} - \frac{1}{6}\cdot\frac{2}{5}\cdot\frac{1}{4}(4x-1)^{5/2} + C$
$= \frac{1}{6}x(4x-1)^{3/2} - \frac{1}{60}(4x-1)^{5/2} + C$

change of variable
$u = \sqrt{4x-1} \quad u^2 = 4x-1 \quad \frac{u^2+1}{4} = x \quad \frac{1}{2}u\,du = dx$

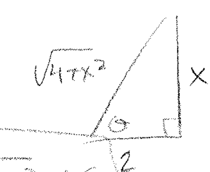
$\int \frac{1}{4}(u^2+1)u \cdot \frac{1}{2}u\,du = \frac{1}{8}\int u^4 + u^2\,du = \frac{1}{8}\cdot\frac{1}{5}u^5 + \frac{1}{8}\cdot\frac{1}{3}u^3 + C$
$\frac{1}{40}(4x-1)^{5/2} + \frac{1}{24}(4x-1)^{3/2} + C$

despite appearances, these are algebraically equivalent

15. $\int \frac{x^3}{\sqrt{4+x^2}}$ $\quad x = 2\tan\theta \quad \sqrt{4+x^2} = 2\sec\theta$
$\quad dx = 2\sec^2\theta \quad x^3 = 8\tan^3\theta$

$\int \frac{8\tan^3\theta \, \cancel{2}\sec^{\cancel{2}}\theta \, d\theta}{\cancel{2}\,\cancel{\sec\theta}} = \int 8\tan^2\theta\cdot(\sec\theta\tan\theta)\,d\theta = 8\int(\sec^2\theta - 1)\sec\theta\tan\theta\,d\theta$
$\quad u = \sec\theta$
$\quad du = \sec\theta\tan\theta$

$8\int u^2 - 1\,du = \frac{8}{3}u^3 - 8u + C =$
$\frac{8}{3}\sec^3\theta - 8\sec\theta + C$
$\frac{8}{3}\left(\frac{\sqrt{4+x^2}}{2}\right)^3 - 8\left(\frac{\sqrt{4+x^2}}{2}\right) + C = \boxed{\frac{1}{3}(4+x^2)^{3/2} - 4\sqrt{4+x^2} + C}$



let's see if it can be done by parts?
$u = x^2 \quad d\cancel{x} = \frac{x}{\sqrt{4+x^2}} = x(4+x^2)^{-1/2} \quad w = 4+x^2$
$\quad \frac{1}{2}dw = \cancel{2}x$

$du = 2x \quad v = \frac{1}{2}(4+x^2)^{1/2}\cdot \cancel{2}$
$= x^2\sqrt{4+x^2} - \int 2x(4+x^2)^{1/2}\,dx = x^2\sqrt{4+x^2} - \frac{2}{3}(4+x^2)^{3/2} + C$

yes we can use this also

16. $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$ can it be done by parts?
we can start it that way anyway

$u = x$ $\quad dv = \frac{x}{\sqrt{1-x^2}} = x(1-x^2)^{-1/2}$ $\quad w = 1-x^2$
$du = dx$ $\quad dw = -2x\,dx$
$v = -(1-x^2)^{1/2}$ $\quad -\frac{1}{2}dw = x\,dx$

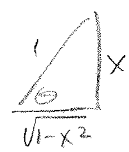
$-x(1-x^2)^{1/2} + \int \sqrt{1-x^2}\,dx$ but now we still have to switch to trig sub for the last term, so let's do that from the beginning

$x = \sin\theta$ $\quad \sqrt{1-x^2} = \cos\theta$
$dx = \cos\theta\,d\theta$ $\quad x^2 = \sin^2\theta$

$\int \frac{\sin^2\theta \cancel{\cos\theta}\,d\theta}{\cancel{\cos\theta}} = \int \sin^2\theta\,d\theta = \int \frac{1}{2}(1-\cos 2\theta)\,d\theta =$

$\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta + C$ to go back to x use identity $\sin 2\theta = 2\sin\theta\cos\theta$

$\frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta + C$



$\frac{1}{2}\arcsin x - \frac{1}{2}x\sqrt{1-x^2} + C$

17. $\int 5^x\,dx = \boxed{\frac{5^x}{\ln 5} + C}$

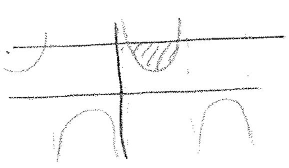
18. $\int x \cdot 4^x\,dx$ $\quad u = x$ $\quad dv = 4^x\,dx$ $\quad \frac{x \cdot 4^x}{\ln 4} - \int \frac{4^x}{\ln 4}\,dx = \boxed{\frac{x \cdot 4^x}{\ln 4} - \frac{4^x}{(\ln 4)^2} + C}$
$du = dx$ $\quad v = \frac{4^x}{\ln 4}$

19. $\int x^2 \operatorname{sech}^2(x^3)\,dx$ $\quad u = x^3$
$du = 3x^2$ $\quad \frac{1}{3}du = x^2$
$\int \frac{1}{3}\operatorname{sech}^2(u)\,du$
$\frac{1}{3}\tanh u + C = \boxed{\frac{1}{3}\tanh(x^3) + C}$

20. $y' = 2 - \frac{\sinh\sqrt{x}}{2\sqrt{x}}$

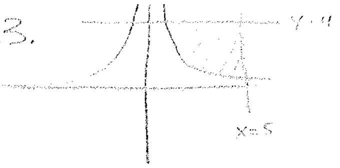
21. $y' = \cosh^2 x + \sinh^2 x$

22.



$\csc x = 2 \Rightarrow \sin x = \frac{1}{2} \Rightarrow x = \pi/6,\ 5\pi/6$

$A = \int_{\pi/6}^{5\pi/6} 2 - \csc x\,dx = 2x + \ln|\csc x + \cot x|\Big|_{\pi/6}^{5\pi/6} =$
$5\pi/3 - \pi/3 + \ln|2+\sqrt{3}| - \ln|2-\sqrt{3}| = \boxed{\frac{4\pi}{3} + \ln\left|\frac{2+\sqrt{3}}{2-\sqrt{3}}\right|}$

23. 

$\frac{1}{x^2} = 4 \Rightarrow 4x^2 = 1 \Rightarrow x^2 = \frac{1}{4} \quad x = \frac{1}{2}$ (ignore $-\frac{1}{2}$ value)

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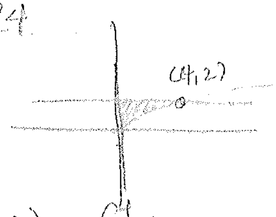
$\int_{1/2}^{5} 4 - \frac{1}{x^2}\,dx = 4x + \frac{1}{x}\Big|_{1/2}^{5} = 20 + \frac{1}{5} - 2 - 2 = 16 + \frac{1}{5} = \boxed{\frac{81}{5}}$

24. 

$y = \sqrt{x} \quad y = 2 \quad x = 0$
using x

$x = y^2 \quad y = 0 \quad x = 0$
using y

a) $\pi \int_0^4 (2)^2 - (\sqrt{x})^2\,dx$ washer $\qquad 2\pi \int_0^2 y(y^2)\,dy$ shell

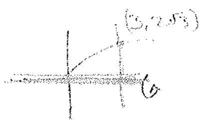
b) $2\pi \int_0^4 x(2 - \sqrt{x})\,dx$ shell $\qquad \pi \int_0^2 (y^2)^2\,dy$ disk

c) $\pi \int_0^4 (\sqrt{x} - 2)^2 - (2-2)^2\,dx$ washer $\qquad 2\pi \int_0^2 (y-2)(y^2)\,dy$ shell

d) $2\pi \int_0^4 (x-6)(2 - \sqrt{x})\,dx$ shell $\qquad \pi \int_0^2 (0-6)^2 - (6-y^2)^2$ washer

25. $y' = \frac{300}{2000} \sinh\left(\frac{x}{2000}\right) = \frac{3}{20} \sinh\left(\frac{x}{2000}\right)$

$S = \int_{-2000}^{2000} \sqrt{1 + \frac{9}{400} \sinh^2\left(\frac{x}{2000}\right)}\,dx$ — this is not readily done by hand. A calculator is good here.
$\approx 3999.897 \Rightarrow 4000$

26. $y' = \frac{1}{\sqrt{x}}$ 

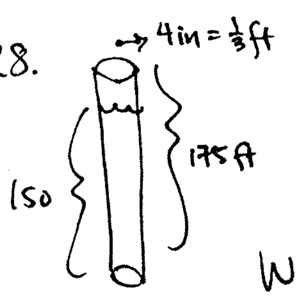
$= \int_0^3 2\sqrt{x}\sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2} = \int_0^3 2\sqrt{x+1}\,dx$

$\int 2u^{1/2}\,du = 2 \cdot \frac{2}{3} u^{3/2} = \frac{4}{3}(x+1)^{3/2}\Big|_0^3 = \frac{4}{3}\left[4^{3/2} - 1^{3/2}\right] =$
$\frac{4}{3}[8-1] = \frac{28}{3}$

$u = x+1$
$du = dx$

27. $F = kx \qquad \frac{50}{6} = \frac{k \cdot \cancel{6}}{\cancel{6}} = \frac{25}{3}$ natural length = 0 $\Rightarrow$ stretch 9 inches

$F = \frac{25}{3}x \qquad W = \int_0^9 \frac{25}{3}x\,dx = \frac{25}{6}x^2\Big|_0^9 = \frac{25}{\cancel{6}_2} \cdot \cancel{81}^{27} = \frac{675}{2}$ inch-pounds

28. 

$F = V \cdot \text{weight}$

weight = 62.4 lbs/ft^3
$V = \pi r^2 h = \pi\left(\frac{1}{3}\right)^2 dy = \frac{\pi}{9}\,dy$

$= \frac{\pi}{9} 62.4\,dy$

end of water line

$W = \int F \cdot \text{distance} = \int_0^{150} 62.4 \cdot \frac{\pi}{9}(175 - y)\,dy = 62.4 \cdot \frac{\pi}{9}[15000]$

distance traveled of a slice at height y

29. $F(x)=\int_1^x \frac{1}{t^2}\,dt \quad F'(x)=\frac{1}{x^2}$

30. $F(x)=\int_3^{x^3} \sin(x^2)\,dx \Rightarrow F'(x)=\sin(x^3)^2\cdot 3x^2 = 3x^2\sin x^6$

31. $f(x)=e^{-x^2} \quad f'(x)=-2xe^{-x^2} \quad f''(x)=4x^2e^{-x^2}-2e^{-x^2}$

$f'''(x)=-8x^3e^{-x^2}+12xe^{-x^2} \qquad f^{IV}(x)=16x^4e^{-x^2}-48x^2e^{-x^2}+12e^{-x^2}$

$f^{V}(x)=-32x^5e^{-x^2}+160x^3e^{-x^2}-120xe^{-x^2}$

for Trapezoidal Rule

$E \le \frac{(b-a)^3}{12n^2}\left[\max|f''(x)|\right]$ $\qquad E=.0001=10^{-4}$

$\frac{(1-(-1))^3}{12n^2}|2| < 10^{-4}$ $\qquad$ max f'' where $f'''=0$

$\cancel{e^{-x^2}}(-8x^3+12x)=0$

$-4x(2x^2-3)=0 \qquad x=0$

$x^2=\frac{3}{2} \quad x=\pm\frac{\sqrt{3}}{\sqrt{2}}$

$\frac{8}{12}|2|\cdot 10^4 < n^2$

$115.47 < n$

$f''(0)=-2$

$n=116$

$f''\left(\frac{\sqrt{3}}{\sqrt{2}}\right)=4e^{-3/2}\approx .89$

$f''\left(-\frac{\sqrt{3}}{\sqrt{2}}\right)=$

for Simpson's Rule

$E \le \frac{(b-a)^5}{180n^4}\left[\max|f^{IV}(x)|\right]$ $\qquad$ max for f^{IV} where $f^{V}=0$

$-8xe^{-x^2}(4x^4-20x^2+15)=0 \qquad x=0$

$\frac{32}{180n^4}|12| < 10^{-4}$

~~$x\approx -2.02$~~ $\quad x\approx -.958 \quad x\approx .958 \quad$ ~~$x\approx 2.02$~~

outside interval

$\frac{32\cdot 12}{180}\cdot 10^4 < n^4$

$f^{IV}(-.96)\approx -7$

$12.0855 < n$

$f^{IV}(0)=12$

$n=14 \leftarrow$ must be even

32. $\frac{1}{b-a}\int_4^9 \frac{1}{\sqrt{x}}\,dx = \frac{1}{5}\int_4^9 x^{-1/2}\,dx = \frac{1}{5}\cdot 2x^{1/2}\Big|_4^9 = \frac{2}{5}[3-2]=\frac{2}{5}$

33. a. $\lim_{x\to 1^+}(x-1)^{\ln x} = 0^0 = L$

$\ln(x-1)^{\ln x}=\ln L \Rightarrow \ln x\cdot\ln(x-1)=\ln L \qquad 0\cdot -\infty$

$$\lim_{x\to1^+}\frac{\ln(x-1)}{\frac{1}{\ln x}} = \frac{\infty}{\infty} \Rightarrow \frac{\frac{1}{x-1}}{-\frac{1}{(\ln x)^2}\cdot\frac{1}{x}} = -\frac{x\cdot(\ln x)^2}{x-1} = \frac{0}{0}$$ 7

$$\Rightarrow \frac{(\ln x)^2 + x\cdot 2\ln x\cdot\frac{1}{\cancel{x}}}{1} = \frac{0+2\cdot0}{1} = 0 \qquad \ln L = 0 \Rightarrow \boxed{L = e^0 = 1}$$

b. $$\lim_{x\to\infty} x e^{-x^2} = \infty\cdot 0 \Rightarrow \lim_{x\to\infty}\frac{x}{e^{x^2}} = \frac{\infty}{\infty} \Rightarrow \frac{1}{2xe^{x^2}} = \frac{1}{\infty} = 0$$

c. $$\lim_{n\to\infty} 1000\left(1+\frac{.09}{n}\right)^n = (1^\infty)1000 = L$$

$$\ln\left[1000\left(1+\frac{.09}{n}\right)^n\right] = \ln L$$

$$\ln 1000 + \ln(1+.09/n)^n = \ln 1000 + n\ln\left(1+\frac{.09}{n}\right) = \ln 1000 + \frac{\ln\left(1+\frac{.09}{n}\right)}{\frac{1}{n}}$$

$$\Rightarrow \ln 1000 + \frac{\frac{1}{1+\frac{.09}{n}}\cdot\frac{-.09}{\cancel{n^2}}}{\cancel{\frac{-1}{n^2}}} \Rightarrow \frac{.09}{1+\frac{.09}{n}}\cdot\frac{n}{n} = \frac{.09n}{n+.09}$$

$$\ln 1000 + \lim_{n\to\infty}\frac{.09n}{n+.09} = \ln 1000 + \frac{.09}{1} = \ln L$$

$$\ln 1000 + \ln e^{.09} = \cancel{\ln}[1000e^{.09}] = \cancel{\ln} L \Rightarrow L = 1000e^{.09}$$

d. $$\lim_{x\to1^+}\left(\frac{2}{\ln x} - \frac{2}{x-1}\right) = \infty - \infty$$

$$= \frac{2(x-1) - 2\ln x}{(\ln x)(x-1)} = \frac{0}{0} \qquad \lim_{x\to1^+}\frac{2-\frac{2}{x}}{\frac{1}{x}(x-1)+\ln x\,(1)}\cdot\frac{x}{x} = \frac{2x-2}{x-1+x\ln x}$$

$$\lim_{x\to1^+} \to \frac{0}{0} \Rightarrow \lim_{x\to1^+}\frac{2}{1+\ln x+1} = \frac{2}{2} = 1$$

34. a. $$\int_1^\infty x^{-1/4}\,dx \to \frac{4}{3}x^{3/4}\Big|_1^b \qquad \lim_{b\to\infty}\frac{4}{3}\left[b^{3/4} - 1^{3/4}\right] = \infty$$
diverges

b. $$\int_0^1 \frac{6}{x-1}\,dx \qquad \lim_{b\to1^-}\int_0^b\frac{6}{x-1}\,dx = 6\ln|x-1|\Big|_0^b$$

$$\lim_{b\to1^-} 6\ln|b-1| - 6\ln|0-1| = \infty \quad \text{diverges}$$

c. $$\int_1^\infty x^2\ln x\,dx \to \lim_{b\to\infty}\int_1^\infty x^2\ln x\,dx \qquad \begin{array}{ll} u = \ln x & dv = x^2dx \\ du = \frac{1}{x} & v = \frac{1}{3}x^3\end{array}$$

$$\lim_{b\to\infty}\frac{1}{3}x^3\ln x - \int_1^b\frac{1}{3}x^2dx = \frac{1}{3}x^3\ln x - \frac{1}{9}x^3\Big|_1^b = \lim_{b\to\infty}\frac{1}{3}b^3\ln b - \frac{1}{9}b^3 - 0 + \frac{1}{9} = \infty - \infty$$

$$\lim_{b\to\infty} \frac{1}{3}b^3 \ln b - \frac{1}{9}b^3 = \lim_{b\to\infty} b^3\left[\frac{1}{3}\ln b - \frac{1}{9}\right] = \frac{\frac{1}{3}\ln b - \frac{1}{9}}{b^{-3}} \Rightarrow$$

$$\lim_{b\to\infty} \frac{\frac{1}{\cancel{3}b}}{-\cancel{3}b^{-4}} = -\frac{b^{\cancel{4}}}{\cancel{b}} = -b^3 = -\infty \quad \text{diverges}$$