Math 2153, Exam #1, Fall 2012 Name KEY

Instructions: Show all work to receive full credit. Answers should be given as exact values unless specifically asked to round. You should do any obvious simplifications, but there is no need to be excessive.

1. Given the points A(2,0,2), B(1,4,-3) and C(5,-3,-2).
   a. Find the vector-valued function that represents the equation of the line connecting points A and B. (10 points)

$\vec{AB} = \langle -1, 4, -5 \rangle$

$\vec{r}(t) = (2-t)\hat{i} + (4t)\hat{j} + (2-5t)\hat{k}$ through A

or

$\vec{r}(t) = (1-t)\hat{i} + (4t+4)\hat{j} + (-5t-3)\hat{k}$ through B

   b. Find the equation of the plane that contains the points A, B and C. (10 points)

$\vec{AC} = \langle 3, -3, -4 \rangle$

$$\vec{AB} \times \vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 4 & -5 \\ 3 & -3 & -4 \end{vmatrix} = (-16-15)\hat{i} - (4+15)\hat{j} + (3-12)\hat{k}$$

$$(-31)\hat{i} - 19\hat{j} - 9\hat{k}$$

$$-31(x-2) - 19y - 9(z-2) = 0$$

2. Rewrite the equation $z = x^2 + y^2 - 2$ into both cylindrical and spherical coordinates. (12 points)

Cylindrical $z = r^2 - 2$

Spherical $\rho\cos\varphi = \rho^2\sin^2\varphi - 2$

3. Determine what kind of quadric surface is given by each equation. (4 points each)

a. $x^2 + y^2 + z^2 + 3x - 4y + 9z = 34$

ellipsoid / Sphere

b. $x^2 + \frac{y^2}{4} = z^2$

cone

c. $z = x^2 + 4y^2$

paraboloid (elliptical)

d. $x = z^2 - y^2 - 4y + 6z + 1$

hyperbolid parabolad

4. Find the equation of the vector-valued function that represents the intersection of the surfaces $x^2 + z^2 = 4, y^2 + z^2 = 4$ using the parameter $z = 2\sin(t)$. Then sketch the graph of the function using 5-10 points. (20 points)

t	x	y	z
0	2	2	0
$\pi/2$	0	0	2
π	-2	-2	0
$3\pi/2$	0	0	-2
2π	2	2	0

$y^2 + z^2 = 4$ $\qquad z = 2\sin t$

$y^2 + 4\sin^2 t = 4$

$y^2 = 4 - 4\sin^2 t = 4(1 - \sin^2 t) = 4\cos^2 t$

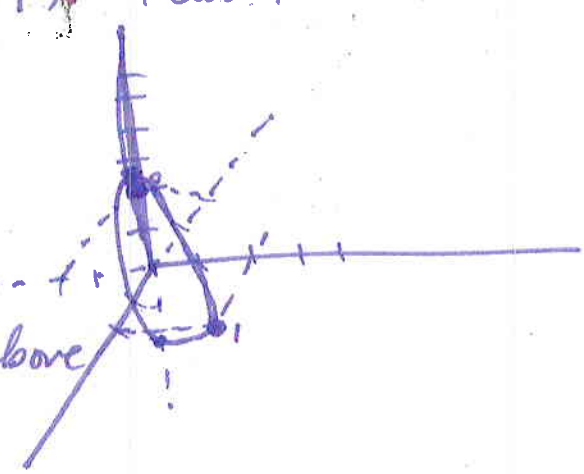
$y = 2\cos t$

$x^2 + z^2 = 4$

$x^2 + 4\sin^2 t = 4$

by same algebra as above

$x = 2\cos t$



$\vec{r}(t) = 2\cos t\,\hat{i} + 2\cos t\,\hat{j} + 2\sin t\,\hat{j}$

5. Find the domain of the vector-valued function $\vec{r}(t) = \sqrt{t}\vec{\imath} + \frac{\cos t}{t^2-2t}\vec{\jmath} + e^{-t}\vec{k}$. (8 points)

$t \geq 0$ $\quad t(t-2)$ $\quad$ all reals

$t \neq 0, t \neq 2$

D: $(0, 2) \cup (2, \infty)$

6. Perform the given operations on the stated functions. (10 points each)

a. Find $\frac{d}{dt}[||\vec{r}(t)||]$ for $\vec{r}(t) = t\sin t\,\hat{\imath} + t\cos t\,\hat{\jmath} + t\hat{k}$.

$\sqrt{t^2\sin^2 t + t^2\cos^2 t + t^2} = \sqrt{t^2(\sin^2 t + \cos^2 t) + t^2} =$

$\sqrt{t^2 + t^2} = \sqrt{2t^2} = \sqrt{2}\,t$

$\frac{d}{dt}[\sqrt{2}\,t] = \boxed{\sqrt{2}}$

b. Find $\frac{d}{dt}[\vec{r}(t) \cdot \vec{s}(t)]$ for $\vec{r}(t) = 7t^2\hat{\imath} + \sin t\,\hat{\jmath} + \cos t\,\hat{k}$, and $\vec{s}(t) = t\hat{\imath} + 6\hat{\jmath} + \tan t\,\hat{k}$

$\vec{r} \cdot \vec{s} = 7t^3 + 6\sin t + \tan t \cdot \cos t$

$\frac{\sin t}{\cancel{\cos t}} \cdot \cancel{\cos t}$

$= 7t^3 + 6\sin t + \sin t = 7t^3 + 7\sin t$

$\frac{d}{dr}[7t^3 + 7\sin t] = \boxed{21t^2 + 7\cos t}$

c. Find $\int_0^1 \vec{r}(t)\,dt$ for $\vec{r}(t) = 2t\hat{\imath} + \frac{1}{t+2}\hat{\jmath} + \frac{1}{t^2+1}\hat{k}$.

$\int_0^1 2t\,dt = t^2\Big|_0^1 = 1$

$\int_0^1 \frac{1}{t+2}dt = \ln|t+2| = \ln 3 - \ln 2 = \ln 3/2$

$\int_0^1 \frac{1}{t^2+1}dt = \arctan t\Big|_0^1 = \pi/4$

$\int_0^1 \vec{r}(t)\,dt = \boxed{\hat{\imath} + \ln 3/2\,\hat{\jmath} + \pi/4\,\hat{k}}$

7. Using the projectile equation $\vec{r}(t) = (\overset{900}{v_0} \cos \overset{45°}{\theta}) t\hat{\imath} + \left[\overset{3}{h} + (\overset{900}{v_0} \sin \overset{45°}{\theta})t - \frac{g}{2}t^2\right]\hat{\jmath}$, determine the maximum height and range of a projectile fired at a height of 3 feet above the ground with an initial velocity of 900 feet per second and at an angle of 45° above the horizontal. You should round your answers to two decimal places. (15 points)

$$\frac{900}{\sqrt{2}} t\,\hat{\imath} + \left[3 + \frac{900}{\sqrt{2}}t - 16t^2\right]\hat{\jmath}$$

$$3 + \frac{900}{\sqrt{2}}t - 16t^2 = 0$$

$t \approx 39.78$ sec. to hit ground

$$\frac{900}{\sqrt{2}}(39.78) = \boxed{25{,}315.84 \text{ feet}}$$ max range.

$$\frac{900}{\sqrt{2}} - 32t = 0 \quad \text{peak}$$

$$t = \frac{\frac{900}{\sqrt{2}}}{32} \approx 19.89 \text{ sec.}$$

$$3 + 900/\sqrt{2}\,(19.89) - 16(19.89)^2 = \boxed{6331.125 \text{ feet.}}$$ max height

8. Find the unit tangent vector and the unit normal vector for the vector-valued function $\vec{r}(t) = 7t\hat{\imath} + 5\sin 2t\,\hat{\jmath} + 5\cos 2t\,\hat{k}$. (20 points)

$r'(t) =$

$$7\hat{\imath} + 10\cos 2t\,\hat{\jmath} - 10\sin 2t\,\hat{k}$$

$$\|\vec{r}\,'(t)\| = \sqrt{49 + 100\cos^2 2t + 100\sin^2 2t} = \sqrt{49+100} = \sqrt{149}$$

$$\boxed{\vec{T}(t) = \frac{1}{\sqrt{149}}\left[7\hat{\imath} + 10\cos 2t\,\hat{\jmath} - 10\sin 2t\,\hat{k}\right]}$$

$$\vec{T}'(t) = \frac{1}{\sqrt{149}}\left[-20\sin 2t\,\hat{\jmath} - 20\cos 2t\,\hat{k}\right]$$

$$\|\vec{T}'(t)\| = \frac{1}{\sqrt{149}}\left[\sqrt{400\sin^2 2t + 400\cos^2 2t}\right] = \frac{\sqrt{400}}{\sqrt{149}} = \frac{20}{\sqrt{149}}$$

$$\vec{N}(t) = -\sin 2t\,\hat{\jmath} - \cos 2t\,\hat{k}$$

9. Find the arc length of the vector-valued function $\vec{r}(t) = 2t^2\hat{\imath} + t^2\hat{\jmath} + \frac{1}{3}t^3\hat{k}$ on the interval [0,3] (10 points)

$$\vec{r}'(t) = 4t\hat{\imath} + 2t\hat{\jmath} + t^2\hat{k}$$

$$\|r'(t)\| = \sqrt{16t^2 + 4t^2 + t^4} = \sqrt{20t^2 + t^4} = \sqrt{t^2(20+t^2)} = t\sqrt{20+t^2}$$

$$\int_0^3 t\sqrt{20+t^2}\,dt \qquad u = 20+t^2, \quad du = 2t\,dt, \quad \tfrac{1}{2}du = t\,dt$$

$$\Rightarrow \int \tfrac{1}{2}u^{1/2}\,du = \tfrac{1}{2}\cdot\tfrac{2}{3}u^{3/2} \Rightarrow \tfrac{1}{3}(20+t^2)^{3/2}\Big|_0^3 =$$

$$\boxed{\tfrac{1}{3}\left[29^{3/2} - 20^{3/2}\right]}$$

10. Find an equation for the curvature of $\vec{r}(t) = t\hat{\imath} + t^2\hat{\jmath} + \frac{1}{3}t^3\hat{k}$. At the point x=1, find the radius of curvature. (14 points)

$$r'(t) = \hat{\imath} + 2t\hat{\jmath} + t^2\hat{k}$$

$$r''(t) = 0\hat{\imath} + 2\hat{\jmath} + 2t\hat{k}$$

$$r' \times r'' = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 2t & t^2 \\ 0 & 2 & 2t \end{vmatrix} =$$

$$(4t^2 - 2t^2)\hat{\imath} - (2t - 0)\hat{\jmath} + (2-0)\hat{k} =$$

$$2t^2\hat{\imath} - 2t\hat{\jmath} + 2\hat{k}$$

$$\|r' \times r''\| = \sqrt{4t^4 + 4t^2 + 4} = 2\sqrt{t^4+t^2+1}$$

$$\|\vec{r}'\| = \sqrt{1 + 4t^2 + t^4}$$

$$\boxed{K = \frac{2\sqrt{t^4+t^2+1}}{(1+4t^2+t^4)^{3/2}}}$$

$$\boxed{x=1 \Rightarrow t=1}$$

$$K(1) = \frac{2\sqrt{3}}{6\sqrt{6}} = \frac{\cancel{2}\,1}{3\cancel{6}\sqrt{2}} = \frac{1}{3\sqrt{2}} = \frac{\sqrt{2}}{6}$$

$$\boxed{R(1) = \frac{6}{\sqrt{2}}} = 3\sqrt{2}$$