Math 2568, Exam #1-Part I, Fall 2013 Name KEY

Instructions: You may **not** use a calculator on this portion of the exam. You should show all work and use exact answers.

1. The system shown below is in vector equation form.

$$x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

   a. Write the equation as a system of linear equations in two variables. (3 points)

$$x_1 - x_2 = 3$$
$$2x_1 + x_2 = 5$$

   b. Write the system as an augmented matrix. (3 points)

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 1 & 5 \end{array}\right]$$

   c. Write the system as a matrix equation. (3 points)

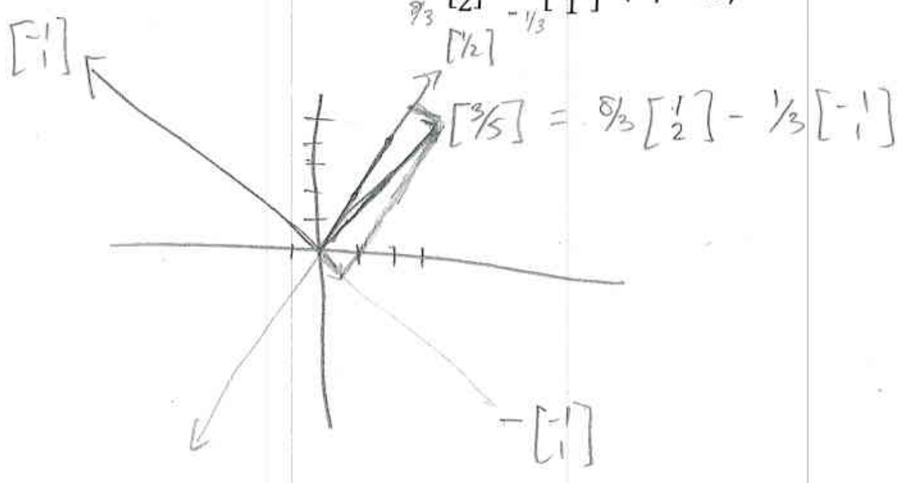
$$\begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

   d. Solve the system, either by reducing the matrix with row operations, or by using an inverse matrix. Write the solution as a column vector. (6 points)

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 2 & 1 & 5 \end{array}\right] \quad -2 \quad 2 \quad -6 \quad -2R_1 + R_2 \Rightarrow R_2 \quad \left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 3 & -1 \end{array}\right] \quad \tfrac{1}{3}R_2 \to R_2$$

$$\left[\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & -1/3 \end{array}\right] \quad R_1 + R_2 \Rightarrow R_1 . \quad \left[\begin{array}{cc|c} 1 & 0 & 8/3 \\ 0 & 1 & -1/3 \end{array}\right] \quad \vec{x} = \begin{bmatrix} 8/3 \\ -1/3 \end{bmatrix}$$

e. Use the solution you obtained and graphically represent it on a graph at the linear combination of the vectors $\begin{bmatrix}1\\2\end{bmatrix}$ and $\begin{bmatrix}-1\\1\end{bmatrix}$. (6 points)



2. Suppose that the matrix $A = \begin{bmatrix}1 & 0 & -5 & 0 & -8 & 3\\ 0 & 1 & 4 & -1 & 0 & 6\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$ is the coefficient matrix of a homogeneous system that is already partially reduced. Finish reducing the system, and then state the solution in parametric form. (10 points)

$8R_3 + R_1 \to R_1$ $\begin{bmatrix}1 & 0 & -5 & 0 & 0 & 3\\ 0 & 1 & 4 & -1 & 0 & 6\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$

$x_1 - 5x_3 + 3x_6 = 0 \Rightarrow$ $x_1 = 5x_3 - 3x_6$

$x_2 + 4x_3 - x_4 + 6x_6 = 0 \Rightarrow$ $x_2 = -4x_3 + x_4 - 6x_6$

x_3 = free $x_3 = x_3$

x_4 = free $x_4 = x_4$

$x_5 = 0$ $x_5 = 0$

x_6 = free. $x_6 = x_6$

$$\vec{x} = x_3\begin{bmatrix}5\\-4\\1\\0\\0\\0\end{bmatrix} + x_4\begin{bmatrix}0\\1\\0\\1\\0\\0\end{bmatrix} + x_6\begin{bmatrix}-3\\-6\\0\\0\\0\\1\end{bmatrix}$$

3. Determine if each statement is True or False. (2 points each)

a. (T) F Every elementary row operation is reversible.

b. T (F) Two equivalent linear systems can have different solutions.

c. T (F) The echelon form of a matrix is unique.

d. T (F) A homogeneous system with free variables has only the trivial solution.

e. (T) F A vector $\vec{x}$ is in the span of a set of vectors $\{\vec{v_1}, \ldots \vec{v_p}\}$ if $\vec{x}$ is a linear combination of the vectors $\vec{v_i}$.

f. (T) F Any set of five real numbers can be represented as a vector in $\mathbf{R}^5$.

g. (T) F If A has a pivot in every row, then $A\vec{x} = \vec{b}$ has a solution for every $\vec{b}$ in $\mathbf{R}^m$.

h. (T) F Both $\begin{bmatrix} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are matrices in echelon form.

i. T (F) If T is a linear transformation mapping $\mathbf{R}^4 \mapsto \mathbf{R}^5$, then it can be represented by a 4x5 matrix.

j. (T) F The product of two matrices A and B is defined in the order AB if A is a mxn matrix and B is an nxp matrix.

k. (T) F A homogeneous equation is always consistent.

l. T (F) $(AB)^T = A^T B^T$.

m. (T) F A linear transformation defined by a 5x6 matrix can be onto, but it cannot be one-to-one.

n. T (F) A matrix given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has a unique solution if $ad - bc = 0$.

o. (T) F If the A matrix has n pivots, then the matrix is invertible. if A is nxn

4. Define the following terms as completely as possible. (5 points)

a. What does it mean for a linear transformation to be **one-to-one**?

for each x in the domain, it maps onto exactly one $\vec{b}$ in the range and each $\vec{b}$ in range comes from only one source in the domain.

b. What is a **projection** transformation? Give an example of it.

a matrix which reduces the # of dimensions from the input for instance $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ projects a vector onto the y-axis.

5. Perform the following matrix operations given $A = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 7 \\ 3 & 6 & -2 \end{bmatrix}$, $C = \begin{bmatrix} 11 & -5 \\ 2 & 4 \\ 0 & -2 \end{bmatrix}$, $D = \begin{bmatrix} 5 & 3 & 0 \\ 2 & -3 & 4 \\ 1 & 0 & 1 \end{bmatrix}$. If the operation is not defined, say so. (4 points each)

a. $B^T + 2C$

$$\begin{bmatrix} 1 & 3 \\ 0 & 6 \\ 7 & -2 \end{bmatrix} + \begin{bmatrix} 22 & -10 \\ 4 & 8 \\ 0 & -4 \end{bmatrix} = \begin{bmatrix} 23 & -7 \\ 4 & 14 \\ 7 & -6 \end{bmatrix}$$

c. AB (2×2) × (2×3) = 2×3

$$\begin{bmatrix} 2+9 & 0+18 & 14-6 \\ 4-3 & 0-6 & 28+2 \end{bmatrix} =$$

$$\begin{bmatrix} 11 & 18 & 8 \\ 1 & -6 & 30 \end{bmatrix}$$

b. BD

(2×3) • (3×3)
= 2×3

$$\begin{bmatrix} 5+0+7 & 3+0+0 & 0+0+7 \\ 15+12-2 & 9-18+0 & 0+24-2 \end{bmatrix}$$

$$\begin{bmatrix} 12 & 3 & 7 \\ 25 & -9 & 22 \end{bmatrix}$$

d. $14A^{-1} + I_2$

$$\left(14 \cdot \frac{1}{-2-12}\right) \begin{bmatrix} -1 & -3 \\ -4 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

-1

$$\begin{bmatrix} 1 & 3 \\ 4 & -2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$$

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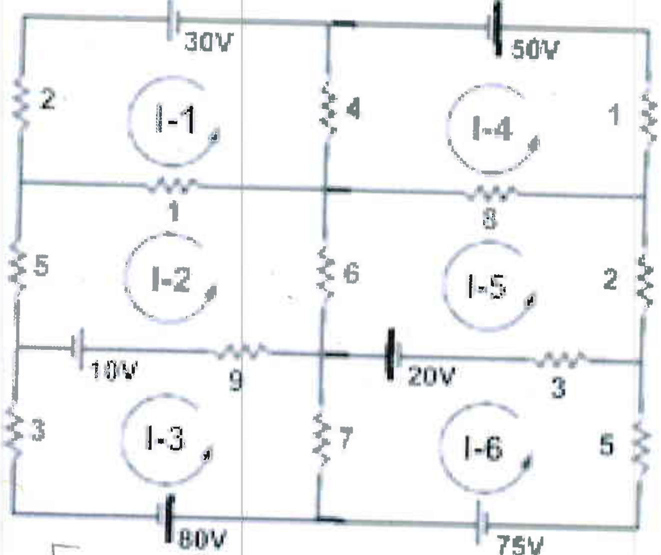
Math 2568, Exam #1-Part II, Fall 2013 Name ______________________

Instructions: You may use a calculator on this portion of the exam, however, I cannot award partial credit where you show no work, and all answers must be justified in some fashion. You should show reduced matrices obtained from the calculator together with their interpretation where appropriate.

1. Based on the graph below, solve the system for the circuit flowing through each loop. The resistance is in Ohms. (10 points) Rounding OK, 2 decimals



$$7I_1 - I_2 - 4I_4 = 30$$
$$-I_1 + 21I_2 - 9I_3 - 6I_5 = 10$$
$$-9I_2 + 19I_3 - 7I_6 = 70$$
$$-4I_1 + 13I_4 - 8I_5 = -50$$
$$-6I_2 - 8I_4 + 19I_5 - 3I_6 = -20$$
$$-7I_3 - 3I_5 + 15I_6 = -55$$

$$\left[\begin{array}{cccccc|c} 7 & -1 & 0 & -4 & 0 & 0 & 30 \\ -1 & 21 & -9 & 0 & -6 & 0 & 10 \\ 0 & -9 & 19 & 0 & 0 & -7 & 70 \\ -4 & 0 & 0 & 13 & -8 & 0 & -50 \\ 0 & -6 & 0 & -8 & 19 & -3 & -20 \\ 0 & 0 & -7 & 0 & -3 & 15 & -55 \end{array}\right]$$

$$\vec{I} = \begin{bmatrix} 1.03 \\ .66 \\ 2.86 \\ -5.87 \\ -3.80 \\ -3.09 \end{bmatrix}$$

2. The vectors $\left\{ \begin{bmatrix} 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -4 \\ -1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ a portion of (or all of) $\mathbb{R}^4$. Determine how many dimensions are in the span of the vectors (i.e. how many linearly independent vectors are there)? (5 points)

rref $\Rightarrow$ $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ dependent, 3 pivots

Spans 3-dimensions of $\mathbb{R}^4$, not all of $\mathbb{R}^4$

3. The matrix $A = \frac{1}{5}\begin{bmatrix} 3 & -4 \\ 4 & 3 \end{bmatrix}$ represents a rotation matrix. Find the angle of rotation. (5 points)

$\cos\theta = 3/5$

$\sin\theta = 4/5$

$\cos^{-1}(3/5) = \theta = .927295$ radians

$\approx 53.13^\circ$

4. Give 6 equivalent statements from the Invertible Matrix Theorem. (6 points)

answers may vary

(1) A is invertible. (2) A^T is invertible

(3) A has n pivots (4) A reduces to the identity

(5) A is one-to-one (6) A is onto

etc.

5. Determine if the linear transformation represented by the matrix $A = \begin{bmatrix} 4 & 2 & 1 \\ -5 & 0 & 6 \\ 3 & -1 & 1 \end{bmatrix}$ is one-to-one, onto, both or neither. (5 points)

reduces to identity

A is one-to-one <u>and</u> onto

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6. Determine whether the transformation given by $T: \vec{x} \in R^3 \mapsto T(\vec{x}) \in R^3$, where T is given by $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - x_2 \\ x_2 + 5x_3 \\ -3x_3 \end{bmatrix}$ is linear or not using the definition (or find a counter example if it is not). If it is linear, find the matrix of the transformation. (12 points)

$$T(\vec{u}+\vec{v}) = \begin{bmatrix} 2(u_1+v_1)-(u_2+v_2) \\ (u_2+v_2)+5(u_3+v_3) \\ -3(u_3+v_3) \end{bmatrix} \quad T(\vec{u})+T(\vec{v}) = \begin{bmatrix} 2u_1-u_2 \\ u_2+5u_3 \\ -3u_3 \end{bmatrix} + \begin{bmatrix} 2v_1-v_2 \\ v_2+5v_3 \\ -3v_3 \end{bmatrix} = \begin{bmatrix} 2u_1+2v_1-u_2-v_2 \\ u_2+v_2+5u_3+5v_3 \\ -3u_3-3v_3 \end{bmatrix} \checkmark$$

$$T(c\vec{u}) = \begin{bmatrix} 2cu_1 - cu_2 \\ cu_2 + 5cu_3 \\ -3cu_3 \end{bmatrix} \quad cT(\vec{u}) = c\begin{bmatrix} 2u_1 - u_2 \\ u_2 + 5u_3 \\ -3u_3 \end{bmatrix} = \begin{bmatrix} 2cu_1 - cu_2 \\ cu_2 + 5cu_3 \\ -3cu_3 \end{bmatrix} \checkmark$$

$$T(\vec{0}) = \vec{0} \checkmark$$

$$A = \begin{bmatrix} 2 & -1 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & -3 \end{bmatrix}$$

7. The matrix $A = \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$ is a shear transformation. Plot the vectors $\vec{u} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$ before the shear transformation and after applying it. You can use the same graph if all the vectors are properly labeled. (10 points)

$$\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 3+8 \\ 0-2 \end{bmatrix} = \begin{bmatrix} 11 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1-8 \\ 0+2 \end{bmatrix} = \begin{bmatrix} -9 \\ 2 \end{bmatrix}$$