Math 1151, Definition of a Derivative Activity, Fall 2014 Name KEY

Instructions: The definition of a limit is shown below. Use the definition to find the derivative of the given functions. Some of the functions may require some identities to simplify.

Definition of a Derivative

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

1. $f(x) = -5x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{-5(x+\Delta x) - (-5x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{-5x} - 5\cancel{\Delta x} + \cancel{5x}}{\cancel{\Delta x}} = -5$$

2. $f(x) = 2 - x^2$

$$f'(x) = \lim_{\Delta x \to 0} \frac{2 - (x+\Delta x)^2 - (2 - x^2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2 - (x^2 + 2x\Delta x + \Delta x^2) - 2 + x^2}{\Delta x} =$$

$$\lim_{\Delta x \to 0} \frac{\cancel{2} - \cancel{x^2} - 2x\Delta x - \Delta x^2 \cancel{-2} + \cancel{x^2}}{\Delta x} = \lim_{\Delta x \to 0} \frac{-2x\Delta x - \Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(-2x - \Delta x)}{\cancel{\Delta x}}$$

$$= -2x$$

3. $f(x) = x^2 + x - 3$

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 + (x+\Delta x) - 3 - (x^2 + x - 3)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{x^2} + 2x\Delta x + \Delta x^2 + \cancel{x} + \Delta x \cancel{-3} - \cancel{x^2} - \cancel{x} + \cancel{3}}{\Delta x}$$

$$\lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(2x + \Delta x + 1)}{\cancel{\Delta x}} = 2x + 1$$

4. $f(x) = x^3 - 12x$

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - 12(x+\Delta x) - x^3 + 12x}{\Delta x} = \lim_{\Delta x \to 0} \frac{\cancel{x^3} + 3x^2\Delta x + 3x\Delta x^2 + \Delta x^3 - \cancel{12x} - 12\Delta x - \cancel{x^3} + \cancel{12x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\cancel{\Delta x}(3x^2 + 3x\Delta x + \Delta x^2 - 12)}{\cancel{\Delta x}} = 3x^2 - 12$$

5. $f(x) = \frac{1}{x-1}$ $\quad f'(x) = \lim_{\Delta x \to 0} \dfrac{\frac{1}{x+\Delta x - 1} - \frac{1}{x-1}}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{\frac{x-1 - (x+\Delta x - 1)}{(x-1)(x+\Delta x - 1)}}{\Delta x} =$

$$\lim_{\Delta x \to 0} \frac{\frac{\cancel{x} \cancel{-1} - \cancel{x} - \cancel{\Delta x} + \cancel{1}}{(x-1)(x+\Delta x - 1)}}{\cancel{\Delta x}} = \lim_{\Delta x \to 0} \frac{-1}{(x-1)(x+\Delta x - 1)} = \frac{-1}{(x-1)^2}$$

6. $f(x) = \frac{4}{\sqrt{x}}$ $f'(x) = \lim_{\Delta x \to 0} \dfrac{\frac{4}{\sqrt{x+\Delta x}} - \frac{4}{\sqrt{x}}}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{\frac{4\sqrt{x} - 4\sqrt{x+\Delta x}}{\sqrt{x}\sqrt{x+\Delta x}}}{\Delta x} \dfrac{(4\sqrt{x}+4\sqrt{x+\Delta x})}{(4\sqrt{x}+4\sqrt{x+\Delta x})}$

$\lim_{\Delta x \to 0} \dfrac{\cancel{16x} - \cancel{16x} - 16\cancel{\Delta x}}{\cancel{\Delta x}\sqrt{x}\sqrt{x+\Delta x}(4\sqrt{x}+4\sqrt{x+\Delta x})} = \lim_{\Delta x \to 0} \dfrac{-\cancel{16}\,4}{\cancel{4}\sqrt{x}\sqrt{x+\Delta x}(\sqrt{x}+\sqrt{x+\Delta x})} = \dfrac{-4}{x(2\sqrt{x})} = -\dfrac{2}{x^{3/2}}$

7. $f(x) = x + \frac{5}{x}$ $\lim_{\Delta x \to 0} \dfrac{\cancel{x} + \Delta x + \frac{5}{x+\Delta x} - \cancel{x} - \frac{5}{x}}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{\cancel{\Delta x}^{1}}{\cancel{\Delta x}} + \dfrac{\cancel{5x} - \cancel{5x} - 5\cancel{\Delta x}}{(x+\Delta x)(x)\cancel{\Delta x}}$

$\lim_{\Delta x \to 0}\left(1 - \dfrac{5}{(x+\Delta x)(x)}\right) = 1 - \dfrac{5}{x^2} = f'(x)$

8. $f(x) = \sqrt{2x-3}$ $\lim_{\Delta x} \dfrac{\sqrt{2x+2\Delta x-3} - \sqrt{2x-3}}{\Delta x} \dfrac{(\sqrt{2x+2\Delta x-3} + \sqrt{2x-3})}{(\sqrt{2x+2\Delta x-3} + \sqrt{2x-3})} =$

$\lim_{\Delta x \to 0} \dfrac{\cancel{2x} + 2\cancel{\Delta x} \cancel{-3} - \cancel{2x} + \cancel{3}}{\cancel{\Delta x}(\sqrt{2x+2\Delta x-3} + \sqrt{2x-3})} \lim_{\Delta x \to 0} \dfrac{2}{\sqrt{2x+2\Delta x-3} + \sqrt{2x-3}} = \dfrac{2}{2\sqrt{2x-3}} = \dfrac{1}{\sqrt{2x-3}}$

$= f'(x)$

9. $f(x) = x^4$ $f'(x) = \lim_{\Delta x \to 0} \dfrac{\cancel{x^4} + 4x^3\Delta x + 6x^2\Delta x^2 + 4x\Delta x^3 + \Delta x^4 - \cancel{x^4}}{\Delta x} =$

$\lim_{\Delta x \to 0} \dfrac{(4x^3 + 6x^2\Delta x + 4x\Delta x^2 + \Delta x^3)\cancel{\Delta x}}{\cancel{\Delta x}} = 4x^3$

10. $f(x) = \sin(x)$ $\lim_{\Delta x \to 0} \dfrac{\sin(x+\Delta x) - \sin(x)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{\sin x(\cos \Delta x) + \cos x \sin \Delta x - \sin x}{\Delta x}$

$\lim_{\Delta x \to 0} \cos \Delta x = 1 \Rightarrow \lim_{\Delta x \to 0} \dfrac{\sin x \cos \Delta x - \sin x}{\Delta x} = 0$

$\lim_{\Delta x \to 0} \cos x \left(\dfrac{\sin \Delta x}{\Delta x}\right) = \cos x = f'(x)$

11. $f(x) = e^x$ $f'(x) = \lim_{\Delta x \to 0} \dfrac{e^{x+\Delta x} - e^x}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{e^x e^{\Delta x} - e^x}{\Delta x} = \lim_{\Delta x \to 0} e^x\left(\dfrac{e^{\Delta x} - 1}{\Delta x}\right)$

It can be shown numerically that $\lim_{\Delta x \to 0} \dfrac{e^{\Delta x} - 1}{\Delta x} = 1$

$\therefore \lim_{\Delta x \to 0} e^x\left(\dfrac{e^{\Delta x} - 1}{\Delta x}\right) = e^x = f'(x)$