Stat 2470 Maximum Likelihood Functions Key

1\. a. $L(p) = p^8(1-p)^{13}$

$L'(p) = 8p^7(1-p)^{13} + p^8(1-p)^{12}\cdot 13(-1)$

$p^7(1-p)^{12}[8(1-p) - 13p] = 0 \qquad 8 \quad 5p = 0 \qquad 25p = 8$

$p = \frac{8}{25} = .32$

this is much more than 1/6 so it does appear that the die is weighted

b. $\frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}} = \frac{\binom{M}{6}\binom{35-M}{2}}{\binom{35}{8}}$ M can be any value between 6 and $(35-2) = 33$

from the Excel data attached the maximum likelihood estimate for M is either 26 or 27
(taking out more decimal places, 26 seems very slightly higher in the 15th decimal place, but this could be a rounding error).

c. $p(x;\mu) = \frac{e^{-\mu}\mu^x}{x!}$

$L(\mu) = \frac{e^{-\mu}\mu^{10}}{10!}\cdot\frac{e^{-\mu}\mu^{15}}{15!}\cdot\frac{e^{-\mu}\mu^{18}}{18!}\cdot\frac{e^{-\mu}\mu^{22}}{22!}\cdot\frac{e^{-\mu}\mu^{19}}{19!}\cdot\frac{e^{-\mu}\mu^{16}}{16!}\cdot\frac{e^{-\mu}\mu^{12}}{12!}$

$= \frac{e^{-7\mu}\mu^{112}}{k}$ where $k = 10!\,15!\,18!\,22!\,19!\,16!\,12!$

$L'(\mu) = \frac{1}{k}\left[-7e^{-7\mu}\mu^{112} + e^{-7\mu}(112)\mu^{111}\right] = 0$

$\frac{1}{k}e^{-7\mu}\mu^{111}[-7\mu + 112] = 0 \Rightarrow 7\mu = 112 \quad \mu = \frac{112}{7}$

$\mu = 16$

(or λ)

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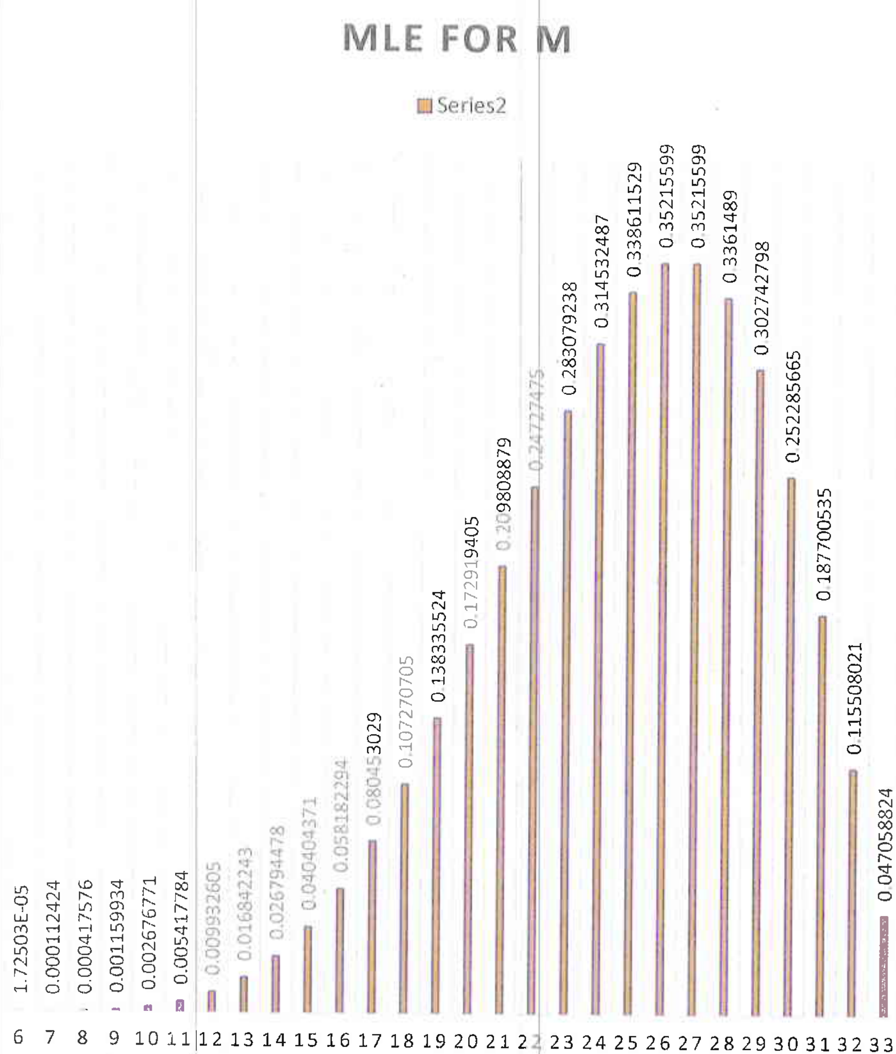
M	probability
6	1.72503E-05
7	0.000112424
8	0.000417576
9	0.001159934
10	0.002676771
11	0.005417784
12	0.009932605
13	0.016842243
14	0.026794478
15	0.040404371
16	0.058182294
17	0.080453029
18	0.107270705
19	0.138335524
20	0.172919405
21	0.209808879
22	0.24727475
23	0.283079238
24	0.314532487
25	0.338611529
26	0.35215599
27	0.35215599
28	0.3361489
29	0.302742798
30	0.252285665
31	0.187700535
32	0.115508021
33	0.047058824



d. $L(\mu,\sigma) = e^{-\frac{(510-\mu)^2}{2\sigma^2}} e^{-\frac{(580-\mu)^2}{2\sigma^2}} e^{-\frac{(430-\mu)^2}{2\sigma^2}} \cdot e^{-\frac{(710-\mu)^2}{2\sigma^2}} \cdot e^{-\frac{(220-\mu)^2}{2\sigma^2}} e^{-\frac{(620-\mu)^2}{2\sigma^2}} \cdot$ ③

$e^{-\frac{(550-\mu)^2}{2\sigma^2}} \cdot e^{-\frac{(490-\mu)^2}{2\sigma^2}} \cdot e^{-\frac{(700-\mu)^2}{2\sigma^2}} \cdot e^{-\frac{(330-\mu)^2}{2\sigma^2}} \cdot \frac{1}{(\sqrt{2\pi})^{10}\sigma^{15}}$

$$L = (\mu,\sigma) = \frac{1}{(2\pi)^5\sigma^{10}} e^{-\frac{1}{2\sigma^2}\left[(510-\mu)^2+(580-\mu)^2+(430-\mu)^2+(710-\mu)^2+(220-\mu)^2+(620-\mu)^2+(550-\mu)^2+(490-\mu)^2+(700-\mu)^2+(330-\mu)^2\right]}$$

$$\frac{\partial L}{\partial \mu} = \frac{e^{-\frac{1}{2\sigma^2}\Sigma(\text{all that})}}{(2\pi)^5\sigma^{10}} 2\left[(510-\mu)+(580-\mu)+(430-\mu)+(710-\mu)+(220-\mu)+(620-\mu)+(550-\mu)+(490-\mu)+(700-\mu)+(330-\mu)\right]$$

$$= 0 \qquad [220+620+550+490+700+330+510+580+430+710 - 10\mu] = 0$$

$5140 = 10\mu \qquad \mu = 514$

$$\frac{\partial L}{\partial \sigma} = e^{-\frac{1}{2\sigma^2}\Sigma(\text{all that})} \cdot [\Sigma(\text{all that})] \cdot \frac{-1}{2} \cdot (-2\sigma^{-3}) \cdot \frac{1}{(2\pi)^5\sigma^{10}}$$

$$+ \frac{1}{(2\pi)^5} e^{-\frac{1}{2\sigma^2}\Sigma(\text{all that})} \sigma^{-11} \cdot (-10) = 0$$

$$\frac{1}{(2\pi)^5} e^{-\frac{1}{2\sigma^2}\Sigma(\text{all that})} \left[\Sigma(\text{all that})\,\sigma^{-13} + \sigma^{-11}(-10)\right] = 0$$

$$\left(\frac{\Sigma(\text{all that})\,\sigma^{-13} = 10\sigma^{-11}}{10}\right)\sigma^{13} \Rightarrow \frac{\Sigma(\text{all that})}{10} = \sigma^2$$

$\sigma^2 = 21784 \qquad \sigma \approx 147.59$

e. $f(x;\lambda) = \lambda e^{-\lambda x}$

$$L(\lambda) = \lambda e^{-22\lambda} \cdot \lambda e^{-34\lambda} \cdot \lambda e^{-16\lambda} \cdot \lambda e^{-25\lambda} \cdot \lambda e^{-29\lambda} \cdot \lambda e^{-45\lambda} \cdot \lambda e^{-32\lambda} \cdot \lambda e^{-11\lambda} \cdot \lambda e^{-27\lambda} = \lambda^9 e^{-241\lambda}$$

$$L'(\lambda) = 9\lambda^8 e^{-241\lambda} + \lambda^9(-241)e^{-241\lambda} = 0$$

$$e^{-241\lambda} \cdot \lambda^8 [9 + \lambda(-241)] = 0 \qquad 241\lambda = 9 \Rightarrow \lambda = \frac{9}{241}$$

4

1. $f(x, \mu, \sigma) = \frac{1}{\sqrt{2\pi}\,\sigma x} e^{\frac{-(\ln x - \mu)^2}{2\sigma^2}}$

$$L(\mu, \sigma) = \frac{1}{\sqrt{2\pi}\sigma \cdot 8} e^{-\frac{(\ln 8-\mu)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma \cdot 4} e^{-\frac{(\ln 4-\mu)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma \cdot 12} e^{-\frac{(\ln 12-\mu)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma \cdot 11} e^{-\frac{(\ln 11-\mu)^2}{2\sigma^2}}$$

$$\cdot \frac{1}{\sqrt{2\pi}\sigma \cdot 9} e^{-\frac{(\ln 9-\mu)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma \cdot 10} e^{-\frac{(\ln 10-\mu)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma \cdot 9} e^{-\frac{(\ln 9-\mu)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma \cdot 8} e^{-\frac{(\ln 8-\mu)^2}{2\sigma^2}}$$

$$\cdot \frac{1}{\sqrt{2\pi}\sigma \cdot 7} e^{-\frac{(\ln 7-\mu)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma \cdot 6} e^{-\frac{(\ln 6-\mu)^2}{2\sigma^2}} \cdot \frac{1}{\sqrt{2\pi}\sigma \cdot 9} \cdot e^{-\frac{(\ln 9-\mu)^2}{2\sigma^2}}$$

$$= \frac{1}{(2\pi)^{11/2}\sigma^{11}\underbrace{(1.138E10)}_{k}} e^{-\frac{1}{2\sigma^2}\left[2(\ln 8-\mu)^2 + (\ln 4-\mu)^2 + (\ln 12-\mu)^2 + (\ln 11-\mu)^2 + 3(\ln 9-\mu)^2 + (\ln 10-\mu)^2 + (\ln 7-\mu)^2 + (\ln 6-\mu)^2\right]}$$

$$\frac{\partial L}{\partial \mu} = \frac{1}{(2\pi)^{11/2}\sigma^{11}k} e^{-\frac{1}{2\sigma^2}[\Sigma(\text{all that})]} \cdot \left(-\frac{1}{2\sigma^2}2\right)\left[2(\ln 8-\mu) + (\ln 4-\mu) + (\ln 12-\mu) + (\ln 11-\mu) + 3(\ln 9-\mu) + (\ln 10-\mu) + (\ln 7-\mu) + (\ln 6-\mu)\right] = 0$$

$$11\mu = 2\ln 8 + \ln 4 + \ln 12 + \ln 11 + 3\ln 9 + \ln 10 + \ln 7 + \ln 6$$

$$\mu \approx 2.096355\ldots$$

$$\frac{\partial L}{\partial \sigma} = \frac{1}{(2\pi)^{11/2}k}(-11)\sigma^{-12} e^{-\frac{1}{2\sigma^2}\Sigma(\text{all that})} + \frac{1}{(2\pi)^{11/2}k}\sigma^{-11} e^{-\frac{1}{2\sigma^2}\Sigma(\text{all that})} \cdot \sigma^{-3}\Sigma(\text{all that})$$

$$= \frac{1}{(2\pi)^{11/2}k} e^{-\frac{1}{2\sigma^2}\Sigma(\text{all that})}\left[-11\sigma^{-12} + \sigma^{-14}\Sigma(\text{all that})\right] = 0 \cdot \frac{\sigma^{14}}{11}$$

$$-\sigma^2 + \frac{1}{11}\Sigma(\text{all that}) = 0 \Rightarrow \sigma^2 = \frac{1}{11}\Sigma(\text{all that})$$

$$\sigma^2 = \frac{1}{11}\left[2(\ln 8 - 2.09636)^2 + (\ln 4 - 2.09636)^2 + (\ln 12 - 2.09636)^2 + (\ln 11 - 2.09636)^2 + 3(\ln 9 - 2.09636)^2 + (\ln 10 - 2.09636)^2 + (\ln 7 - 2.09636)^2 + (\ln 6 - 2.09636)^2\right]$$

$\sigma^2 \approx .085011 \qquad \sigma \approx .2915669\ldots$

2. for answers, see above in #1

3.a $f(x, \alpha) = \alpha^2 x e^{-\alpha x}, \; x \geq 0$

$$L(\alpha) = \alpha^2(1)e^{-\alpha} \cdot \alpha^2(1.4)e^{-1.4\alpha} \cdot \alpha^2(0.6)e^{-.6\alpha} \cdot \alpha^2(0.5)e^{-.5\alpha} \cdot \alpha^2(1.1)e^{-1.1\alpha} \cdot \alpha^2(0.2)e^{-.2\alpha}$$

$$\cdot \alpha^2(0.3)e^{-.3\alpha} \cdot \alpha^2(0.2)e^{-.2\alpha} \cdot \alpha^2(0.9)e^{-.9\alpha}$$

3a cont'd

$$L(\alpha) = \alpha^{18}(\underbrace{.0049896}_{k})e^{-6.2\alpha}$$

$$L'(\alpha) = 18\alpha^{17}ke^{-6.2\alpha} + \alpha^{18}k(-6.2)e^{-6.2\alpha} = 0$$

$$k\alpha^{17}e^{-6.2\alpha}[18 - 6.2\alpha] = 0 \qquad \frac{6.2\alpha = 18}{6.2} \qquad \alpha \approx 2.90$$

3b. $f(x,\alpha) = \frac{2\sqrt{\alpha}}{\pi(1+\alpha x^2)}$ $x \geq 0$

$$L(\alpha) = \frac{2\sqrt{\alpha}}{\pi(1+\alpha(.1)^2)} \cdot \frac{2\sqrt{\alpha}}{\pi(1+\alpha(.5)^2)} \cdot \frac{2\sqrt{\alpha}}{\pi(1+\alpha(.6)^2)} \cdot \frac{2\sqrt{\alpha}}{\pi(1+\alpha(.9)^2)} \cdot \frac{2\sqrt{\alpha}}{\pi(1+\alpha)} \cdot \frac{2\sqrt{\alpha}}{\pi(1+1.4^2\alpha)} \cdot \frac{2\sqrt{\alpha}}{\pi(1+1.7^2\alpha)}$$

$$\cdot \frac{2\sqrt{\alpha}}{\pi(1+2.1^2\alpha)} \cdot \frac{2\sqrt{\alpha}}{\pi(1+\alpha(3.2)^2)} \cdot \frac{2\sqrt{\alpha}}{\pi(1+5.6^2\alpha)}$$

$$= \frac{1024\alpha^5}{\pi^{10}(1+.01\alpha)(1+.25\alpha)(1+.36\alpha)(1+.81\alpha)(1+\alpha)(1+1.96\alpha)(1+2.89\alpha)(1+4.41\alpha)(1+10.24\alpha)\cdot(1+31.36\alpha)}$$

$$\ln(L(\alpha)) = \ln\left(\frac{1024}{\pi^{10}}\right) + \ln\alpha^5 - \ln(1+.01\alpha) - \ln(1+.25\alpha) - \ln(1+.36\alpha) - \ln(1+.81\alpha) -$$
$$\ln(1+\alpha) - \ln(1+1.96\alpha) - \ln(1+2.89\alpha) - \ln(1+4.41\alpha) - \ln(1+10.24\alpha)$$
$$- \ln(1+31.36\alpha)$$

$$\frac{1}{L(\alpha)}\cdot L'(\alpha) = \frac{1}{\alpha^5}\cdot 5\alpha^4 - \frac{1}{1+.01\alpha}(.01) - \frac{.25}{1+.25\alpha} - \frac{.36}{1+.36\alpha} - \frac{.81}{1+.81\alpha} - \frac{1}{1+\alpha} - \frac{1.96}{1+1.96\alpha}$$
$$- \frac{2.89}{1+2.89\alpha} - \frac{4.41}{1+4.41\alpha} - \frac{10.24}{1+10.24\alpha} - \frac{31.36}{1+31.36\alpha}$$

$$L'(\alpha) = \frac{1024}{\pi^{10}}\left[\frac{5}{\alpha} - \frac{.01}{1+.01\alpha} - \frac{.25}{1+.25\alpha} - \frac{.36}{1+.36\alpha} - \frac{.81}{1+.81\alpha} - \frac{1}{1+\alpha} - \frac{1.96}{1+1.96\alpha} - \frac{2.89}{1+2.89\alpha} - \frac{4.41}{1+4.41\alpha} - \frac{10.24}{1+10.24\alpha} - \frac{31.36}{1+31.36\alpha}\right]$$

$$\cdot\left[\frac{\alpha^5}{(1+.01\alpha)(1+.25\alpha)(1+.36\alpha)(1+.81\alpha)(1+\alpha)(1+1.96\alpha)(1+2.89\alpha)(1+4.41\alpha)(1+10.24\alpha)(1+31.36\alpha)}\right]$$

$5(1+.01\alpha)(1+.25\alpha)(1+.36\alpha)(1+.81\alpha)(1+\alpha)(1+1.96\alpha)(1+2.89\alpha)(1+4.41\alpha)(1+10.24\alpha)(1+31.36\alpha) -$
$.01(1+.25\alpha)(1+.36\alpha)(1+.81\alpha)(1+\alpha)(1+1.96\alpha)(1+2.89\alpha)(1+4.41\alpha)(1+10.24\alpha)(1+31.36\alpha) - .25(1+.01\alpha)\cdot$
$(1+.36\alpha)(1+.81\alpha)(1+\alpha)(1+1.96\alpha)(1+2.89\alpha)(1+4.41\alpha)(1+10.24\alpha)(1+31.36\alpha) - .36(1+.01\alpha)(1+.25\alpha)(1+.81\alpha)$
$(1+\alpha)(1+2.89\alpha)(1+4.41\alpha)(1+10.24\alpha)(1+31.36\alpha) - .81(1+.01\alpha)(1+.25\alpha)(1+.36\alpha)(1+\alpha)(1+1.96\alpha)(1+2.89\alpha)$
$(1+4.41\alpha)(1+10.24\alpha)(1+31.36\alpha) - 1(1+.01\alpha)(1+.25\alpha)(1+.36\alpha)(1+.81\alpha)(1+1.96\alpha)(1+2.89\alpha)(1+4.41\alpha)(1+10.24\alpha)$
$(1+31.36\alpha) - 1.96(1+.01\alpha)(1+.25\alpha)(1+.36\alpha)(1+.81\alpha)(1+\alpha)(1+2.89\alpha)(1+4.41\alpha)(1+10.24\alpha)(1+31.36\alpha) -$
$2.89(1+.01\alpha)(1+.25\alpha)(1+.36\alpha)(1+.81\alpha)(1+\alpha)(1+1.96\alpha)(1+4.41\alpha)(1+10.24\alpha)(1+31.36\alpha) - 4.41(1+.01\alpha)$
$(1+.25\alpha)(1+.36\alpha)(1+.81\alpha)(1+\alpha)(1+1.96\alpha)(1+2.89\alpha)(1+10.24\alpha)(1+31.36\alpha) - 10.24(1+.01\alpha)(1+.25\alpha)$
$(1+.36\alpha)(1+.81\alpha)(1+\alpha)(1+1.96\alpha)(1+2.89\alpha)(1+4.41\alpha)(1+31.36\alpha) - 31.36(1+.01\alpha)(1+.25\alpha)(1+.36\alpha)$
$(1+.81\alpha)(1+\alpha)(1+1.96\alpha)(1+2.89\alpha)(1+4.41\alpha)(1+10.24\alpha) = 0$

3b cont'd

$$\frac{dL}{d\alpha} = -.009349x^4(x^{10}+88.1799x^9+636.852x^8+1585.82x^7+1438.88x^6$$
$$-1.40226\times10^{-10}x^5-727.446x^4-411.091x^3-88.1364x^2-7.2902x$$
$$-.171003)/[(x+100)^2(x+4)^2(x+.277778)^2(x+1.23457)^2(x+1)^2$$
$$(x+.510204)^2(x+.346021)^2(x+.226757)^2(x+.097656)^2(x+.031888)^2]$$

Zeros at -80.517, -3.54912, -2.17636, -1.12086, -7.04509, $-.413194$, $-.262785$, $-.117825$, $-.03809$, 0, $.71458$

negatives and 0' excluded for $\sqrt{a}$ term.

$\therefore \alpha = .71458$

3c. $f(x;\alpha,\beta) = \dfrac{4\alpha\sqrt{\beta}}{\pi(\alpha^2+\beta x^4)}$, $x \geq 0$

$$L(\alpha,\beta) = \frac{4\alpha\sqrt{\beta}}{\pi(\alpha^2+\beta)}\cdot\frac{4\alpha\sqrt{\beta}}{\pi(\alpha^2+256\beta)}\cdot\left[\frac{4\alpha\sqrt{\beta}}{\pi(\alpha^2+625\beta)}\right]^2\cdot\frac{4\alpha\sqrt{\beta}}{\pi(\alpha^2+2401\beta)}\cdot\frac{4\alpha\sqrt{\beta}}{\pi(\alpha^2+4096\beta)}\cdot\frac{4\alpha\sqrt{\beta}}{\pi(\alpha^2+10000\beta)}$$
$$\cdot\frac{4\alpha\sqrt{\beta}}{\pi(\alpha^2+28561\beta)} =$$

$$\frac{65536\alpha^8\beta^4}{\pi^8(\alpha^2+\beta)(\alpha^2+256\beta)(\alpha^2+625\beta)(\alpha^2+2401\beta)(\alpha^2+4096\beta)(\alpha^2+10{,}000\beta)(\alpha^2+28561\beta)}$$

$$\frac{\partial L}{\partial \alpha} = -131072\alpha^7(4\alpha^{14}+137195\alpha^{12}\beta+114343282\,5\alpha^{10}\beta^2+2375163360545$$
$$\alpha^8\beta^3-14844771003406\,25\alpha^6\beta^4-4951595366512405536\alpha^4\beta^5$$
$$-1356015992761433600000\alpha^2\beta^6-179765145763\,84\times10^8\beta^7)\beta^4/$$
$$\pi^8(\alpha^2+28561\beta)^2(\alpha^2+10000\beta)^2(\alpha^2+4096\beta)^2(\alpha^2+2401\beta)^2(\alpha^2+625\beta)^3$$
$$(\alpha^2+256\beta)^2(\alpha^2+\beta)^2$$

3c. contd

$$\frac{\partial L}{\partial \beta} = -65536\beta^3(17976514576384\times10^8\beta^7 + 1356015992761433600000\beta^6\alpha^2$$
$$+ 4951595366512405536\beta^5\alpha^4 + 1484477100340625\beta^4\alpha^6$$
$$- 2375163360545\beta^3\alpha^8 - 1143432825\beta^2\alpha^{10} - 137195\beta\alpha^{12} - 4\alpha^{14})\alpha^8$$
$$/[\pi^8(28561\beta+\alpha^2)^2(10000\beta+\alpha^2)^2(4096\beta+\alpha^2)^2(2401\beta+\alpha^2)^2(625\beta+\alpha^2)^3$$
$$(256\beta+\alpha^2)^2(\beta+\alpha^2)^2$$

find the solution between 0, and 2.

on the attached graph $x=\alpha$ and $y=\beta$

given the coefficients the solution is extremely small.