MAT 192, Exam #3, Fall 2016 Name KEY

**Instructions**: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Find the first five terms of the sequence. (3 points each)

a. $a_n = \frac{3^n}{1+2^n}$

$$\left\{\frac{1}{2}, \frac{3}{3}, \frac{9}{5}, \frac{27}{9}, \frac{81}{17}, \ldots\right\}$$

b. $a_1 = 6, a_{n+1} = \frac{a_n}{n}$

$$\left\{6, \frac{6}{2}, \frac{3}{3}, \frac{1}{4}, \frac{1}{20}, \cdots\right\}$$

2. Write the sequence $\left\{\frac{1}{2}, -\frac{4}{3}, \frac{9}{4}, -\frac{16}{5}, \frac{25}{6}, \ldots\right\}$ as a formula for $a_n$ in terms of $n$. (4 points)

$$a_n = \frac{(-1)^{n+1} n^2}{n+1}$$

(n=1) start

or

$$a_n = \frac{(-1)^n (n+1)^2}{(n+2)}$$

(n=0 start)

3. Determine whether the sequence converges or diverges. If it converges, what does it converge to? (3 points each)

a. $a_n = \frac{n^3}{n+1}$

diverges

$$\lim_{n\to\infty} \frac{n^3}{n+1} = \infty$$

c. $a_n = \ln(n+1) - \ln n$

$$\lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right) =$$

$$\ln\left(\lim_{n\to\infty} \frac{n+1}{n}\right) = \ln 1 = 0$$

converges

b. $a_n = e^{2n/(n+2)}$

$$\lim_{n\to\infty} e^{\frac{2n}{n+2}} = \lim_{n\to\infty} e^{2-\frac{4}{n+2}} =$$

$$\begin{array}{r} 2 \\ n+2\overline{)2n} \\ -2n\mp4 \\ \hline -4 \end{array}$$

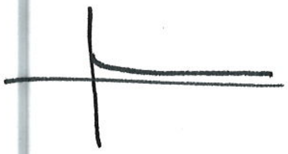
$$\lim_{n\to\infty} e^2 \cdot e^{-4/n+2} = e^2 \cdot e^{\lim_{n\to\infty} \frac{-4}{n+2}} =$$

$$e^2 \cdot e^0 = \boxed{e^2}$$

Converges

4. Use a graph of the sequence $a_n = \sqrt{\frac{3+2n^2}{8n^2+n}}$ to determine the convergence (or divergence) of the sequence. (4 points)

$$\lim_{n\to\infty} \sqrt{\frac{3+2n^2}{8n^2+n}} = \sqrt{\lim_{n\to\infty} \frac{3+2n^2}{8n^2+n}} =$$

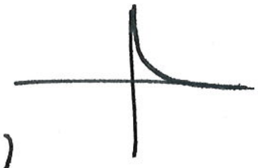


$$\sqrt{\frac{1}{4}} = 1/2$$

Converges

5. Is the sequence $a_n = \frac{n}{n^2+1}$ increasing or decreasing? Is it monotonic (after some finite value of $n$)? Is the sequence bounded? Explain your reasoning. (4 points)

decreasing, monotonic
bounded above by 1/2, $(n=1)$
bounded below by 0 (always positive)



$$f(x) = \frac{x}{x^2+1}$$

$$f'(x) = \frac{x^2+1-x(x\cdot 2)}{(x^2+1)^2} = \frac{-x^2+1}{(x^2+1)^2}$$

neg for $x > 1$ so decreasing
never changes sign again

6. Write $2.5\overline{16}$ as the ratio of integers using a geometric series. (3 points)

$$2.5 + \frac{16}{10}\sum_{n=1}^{\infty} \frac{1}{(100)^n}$$

$r = \frac{1}{100}$ $a = \frac{16}{10}$

$$\frac{16}{10}\sum_{n=1}^{\infty} \frac{1}{100^n} = \frac{16/1000}{1-\frac{1}{100}} = \frac{16/1000}{99/100} =$$

$$\frac{16}{1000} \cdot \frac{100}{99} = \frac{16}{990}$$

$$\frac{5}{2} + \frac{16}{990} =$$

$$\frac{4950}{1980} + \frac{32}{1980} = \frac{4982}{1980} = \frac{2491}{990}$$

7. Find the sum of the series, if it exists. (4 points each)

a. $\sum_{n=1}^{\infty}\frac{3^{n+1}}{\pi^n} = \sum_{n=1}^{\infty} 3\left(\frac{3}{\pi}\right)^n$ geometric

$$\frac{3\left(\frac{3}{\pi}\right)}{1-\frac{3}{\pi}} = \frac{9/\pi}{\frac{\pi-3}{\pi}} = \frac{9}{\cancel{\pi}}\cdot\frac{\cancel{\pi}}{\pi-3} = \frac{9}{\pi-3}$$

c. $\sum_{n=1}^{\infty}\left(e^{1/n} - e^{1/(n+1)}\right)$

telescoping

$$e^1 - \lim_{n\to\infty} e^{\frac{1}{n+1}} = \boxed{e-1}$$

b. $\sum_{n=3}^{\infty}\left(\frac{3}{n}\right)^4 = \sum_{n=3}^{\infty}\frac{81}{n^4} = \sum_{n=1}^{\infty}\frac{81}{n^4} - 81 - \frac{81}{16}$

$$= \frac{81\pi^4}{90} - 81 - \frac{81}{16} = \boxed{\frac{9\pi^4}{10} - \frac{1377}{16}}$$

8. Use the integral test to determine the convergence of the series. If it does converge, estimate the error after 10 terms. (5 points each)

a. $\sum_{n=1}^{\infty}\frac{1}{n^2+6n+13}$

$$\int_{10}^{\infty}\frac{1}{n^2+6n+9+4}dn =$$

$$\int_{10}^{\infty}\frac{1}{(x+3)^2+4}dx$$

$$\frac{1}{2}\arctan\left(\frac{x+3}{2}\right)\Big|_{10}^{\infty}$$

$$\frac{1}{2}\frac{\pi}{2} - \frac{1}{2}\arctan\left(\frac{13}{2}\right) \approx .07632\ldots$$

Converges

b. $\sum_{n=2}^{\infty}\frac{1}{n\ln n}$

$u = \ln n$

$du = \frac{1}{n}dn$

$$\int_2^{\infty}\frac{1}{u}du \Rightarrow \ln u$$

$$\ln(\ln n)\Big|_2^{\infty} = \lim_{b\to\infty}\ln(\ln b) - \ln(\ln 2)$$

diverges

9. Use a comparison test to determine the convergence or divergence of the series. (4 points each)

a. $\sum_{n=0}^{\infty}\frac{9^n}{3+10^n}$

$b_n = \frac{9^n}{10^n}$

converges by geometric series $\frac{9}{10} < 1$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \frac{\lim_{n\to\infty}\cancel{9^n}}{3+10^n}\cdot\frac{10^n}{\cancel{9^n}}$$

$$\lim_{n\to\infty}\frac{10^n}{3+10^n} = 1$$ Converges

c. $\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{3n^4-1}}$

$b_n = \frac{1}{n^{4/3}} = \frac{1}{\sqrt[3]{n^4}}$ converges by p series $p>1$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{\sqrt[3]{3n^4-1}}\cdot\frac{\sqrt[3]{n^4}}{1} =$$

$$\lim_{n\to\infty}\sqrt[3]{\frac{n^4}{3n^4+1}} = \sqrt[3]{\frac{1}{3}}$$ converges

b. $\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$ compare to $b_n = \frac{1}{n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2-5n}{n^3+n+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n^3-5n^2}{n^3+n+1} = 1$$

$\frac{1}{n}$ diverges by p-series/integral test (harmonic)

So this series also diverges

10. Determine if the series converges or diverges. If it converges, does it converge absolutely or conditionally? (4 points each)

a. $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{2n+3}$

Converges conditionally

$$\lim_{n\to\infty} \left|\frac{(-1)^n\sqrt{n}}{2n+3}\right| = \lim_{n\to\infty} \frac{1}{2\sqrt{n}+\frac{3}{\sqrt{n}}} = 0$$

but by comparison w/ $\frac{1}{\sqrt{n}}$

$$\lim_{n\to\infty} \frac{\sqrt{n}}{2n+3} \cdot \frac{\sqrt{n}}{1} = \frac{n}{2n+3} = \frac{1}{2}$$

$\frac{1}{\sqrt{n}}$ diverges by p-series test

So this diverges w/o $(-1)^n$

b. $\sum_{n=1}^{\infty} (-1)^n e^{-n^2} n!$

$$\lim_{n\to\infty} \left|\frac{\cancel{(-1)^{n+1}}(n+1)!}{e^{(n+1)^2}} \cdot \frac{e^{n^2}}{\cancel{(-1)^n}n!}\right| =$$

$$\lim_{n\to\infty} \left|\frac{(n+1)\cancel{n!}\,e^{n^2}}{\cancel{n!}\,e^{n^2+2n+1}}\right| = \lim_{n\to\infty} \left|\frac{\cancel{e^{n^2}}(n+1)}{\cancel{e^{n^2}}\cdot e^{2n+1}}\right| = 0$$

Converges absolutely

11. How many terms are needed to estimate the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n n!}$ to within $E \leq 10^{-6}$? [Hint: You may need to do this numerically in your calculator.] (4 points)

$$|a_n| = \frac{1}{3^n n!}$$

$3^N N! \geq 10^6$

$N=6 \Rightarrow 524,880$

$N=7 \Rightarrow 1.1E7$ error term

<u>need 7 terms (to n=6)</u>

counting $n=0$

12. Use the root or ratio test to determine if the series converges or diverges. (4 points each)

a. $\sum_{n=1}^{\infty} \frac{3^{n^2}}{n!}$

$$\lim_{n\to\infty}\left|\frac{3^{(n+1)^2}}{(n+1)!}\cdot\frac{n!}{3^{n^2}}\right| =$$

$$\lim_{n\to\infty}\left|\frac{\cancel{3^{n^2}}\,3^{2n+1}\,\cancel{n!}}{(n+1)\cancel{n!}\,\cancel{3^{n^2}}}\right| =$$

$$\lim_{n\to\infty}\left|\frac{3^{2n+1}}{n+1}\right| = \infty$$

diverges

b. $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

$$\lim_{n\to\infty}\sqrt[n]{\left(1+\frac{1}{n}\right)^{n^2}} =$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} = e > 1$$

diverges

13. For each series select an appropriate test to determine convergence or divergence of the series. (5 points each)

a. $\sum_{n=1}^{\infty} \frac{n^2 2^{n-1}}{(-5)^n}$

ratio or root test

(converges)

d. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

integral test

diverges

b. $\sum_{n=1}^{\infty} \frac{\sin 2n}{1+2^n}$

Comparison test

(limit)

Converges

e. $\sum_{k=2}^{\infty} \frac{k \ln k}{(k+1)^3}$

Comparison test

(direct)

Converges

c. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh n}$

alternating

Series test

converges

14. Determine the interval of convergence of the radius of convergence for each power series. Be sure to check the endpoints of each interval. (5 points each)

a. $\sum_{n=2}^{\infty}\frac{(-1)^n x^n}{4^n \ln n}$

$$\lim_{n\to\infty}\left|\frac{x^{\cancel{n}+1}}{4^{\cancel{n}+1}\ln(n+1)}\cdot\frac{\cancel{4^n}\ln(n)}{\cancel{x^n}}\right| =$$

$\frac{x}{4} < 1$

$-1 < x/4 < 1$

$-4 < x < 4$

$R=4$ $(-4,4) \Rightarrow \boxed{(-4,4]}$

$x=-4$

$\sum \frac{1}{\ln n}$ diverges

$\sum \frac{(-1)^n}{\ln n}$ converges

b. $\sum_{n=1}^{\infty}\frac{n(2x-1)^n}{5^n}$

$$\lim_{n\to\infty}\left|\frac{(\cancel{n}+1)(2x-1)^{\cancel{n}+1}}{5^{\cancel{n}+1}}\cdot\frac{\cancel{5^n}}{\cancel{n}\,\cancel{(2x-1)^n}}\right| =$$

$\lim_{n\to\infty}\left|\frac{2x-1}{5}\right| < 1$

$x=-2$

$\sum \frac{n(-5)^n}{5^n}$ diverges

$\sum \frac{n\,5^n}{5^n}$ diverges

$-1 < \frac{2x-1}{5} < 1$

$-\frac{5}{2} < \frac{2x-1}{2} < \frac{5}{2}$ $R=5/2$

$-\frac{5}{2} < x - 1/2 < 5/2$

$+1/2$ $+1/2$ $+1/2$

$-2 < x < 3$

$\boxed{(-2,3)}$

15. Find a power series for the function. (4 points each)

a. $f(x) = \frac{2}{3-x} = \frac{2/3}{1-1/3x}$

$a = 2/3$, $r = 1/3x = \frac{x}{3}$

$$f(x) = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)\left(\frac{x}{3}\right)^n$$

b. $f(x) = \frac{x}{(1+4x^2)^2}$

$a = x$ $r = -4x^2$

$\sum_{n=0}^{\infty}(n+1)x^n = \frac{1}{(1-r)^2}$

$$f(x) = x\sum_{n=0}^{\infty}(-4x^2)^n(n+1) =$$

$$\sum_{n=0}^{\infty}(-4)^n(n+1)x^{2n+1}$$

16. Use a power series to evaluate $\int \frac{t}{1+t^3}dt$. (5 points)

$r = -t^3$ $a = t$

$$\int \sum_{n=0}^{\infty} t(-t^3)^n\,dt = \sum_{n=0}^{\infty}(-1)^n\int t^{3n+1}\,dt =$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n t^{3n+2}}{3n+2} + C$$

17. Rewrite $x^4 + 3x^2 + 1$ as a Taylor series centered at $c = 1$. (4 points)

| n | n! | $f^{(n)}(x)$ | $f^{(n)}(c)$ | $(x-c)^n$ | $\frac{f^{(n)}(c)}{n!}(x-c)^n$ |
|---|---|---|---|---|---|
| 0 | 1 | $x^4+3x^2+1$ | 5 | 1 | 5 |
| 1 | 1 | $4x^3+6x$ | 10 | $(x-1)$ | $10(x-1)$ |
| 2 | 2 | $12x^2+6$ | 18 | $(x-1)^2$ | $9(x-1)^2$ |
| 3 | 6 | $24x$ | 24 | $(x-1)^3$ | $4(x-1)^3$ |
| 4 | 24 | 24 | 24 | $(x-1)^4$ | $(x-1)^4$ |
| 5 | 120 | 0 | 0 | — | |
| 6 | 720 | | | | |

$P_n(x) = (x-1)^4 + 4(x-1)^3 + 9(x-1)^2 + 10(x-1) + 5$

18. Determine the number of terms needed to estimate $f(x) = e^{2x}$ at $x = 3.5$ with a power series centered at $c = 3$ to within $E \leq 10^{-4}$. (5 points)

| n | n! | $f^{(n)}(x)$ | $f^{(n)}(c)$ | $(x-c)^n$ | $\frac{f^{(n)}(c)}{n!}(x-c)^n$ | |
|---|---|---|---|---|---|---|
| 0 | 1 | $e^{2x}$ | $e^6$ | 1 | $e^6$ | |
| 1 | 1 | $2e^{2x}$ | $2e^6$ | $(x-3)$ | $2e^6(x-3)$ | $\approx 403$ |
| 2 | 2 | $4e^{2x}$ | $4e^6$ | $(x-3)^2$ | $2e^6(x-3)^2$ | $\approx 201$ |
| 3 | 6 | $8e^{2x}$ | $8e^6$ | $(x-3)^3$ | $\frac{4}{3}e^6(x-3)^3$ | $\approx 67$ |
| 4 | 24 | $16e^{2x}$ | $16e^6$ | $(x-3)^4$ | $\frac{2}{3}e^6(x-3)^4$ | $\approx 16.8$ |
| 5 | 120 | $32e^{2x}$ | $32e^6$ | $(x-3)^5$ | $\frac{4}{15}e^6(x-3)^5$ | $\approx 3.36$ |
| 6 | 720 | $64e^{2x}$ | $64e^6$ | $(x-3)^6$ | $\frac{4}{45}e^6(x-3)^6$ | $\approx .56$ |
| 7 | 5040 | $128e^{2x}$ | $128e^6$ | $(x-3)^7$ | $\frac{8}{315}e^6(x-3)^7$ | $\approx .08$ |
| 8 | 40320 | $256e^{2x}$ | $256e^6$ | $(x-3)^8$ | $\frac{2}{315}e^6(x-3)^8$ | $\approx .01$ |
| 9 | | $512e^{2x}$ | $512e^6$ | $(x-3)^9$ | $\frac{4}{3835}e^6(x-3)^9$ | $\approx .001$ |
| 10 | | $1024e^{2x}$ | $1024e^6$ | $(x-3)^{10}$ | $\frac{4}{38350}e^7(1/2)^{10}$ | $\approx .000117$ |

n=0+ next 10 terms: n = 11 terms are needed   error falls below $10^{-4}$ w/ n=11

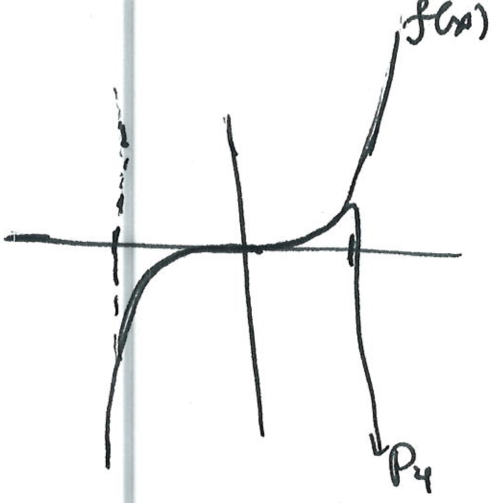
19. Find a Maclaurin series for $f(x) = x^2 \ln(1 + x^3)$. Use the table of Maclaurin series included at the end of the exam. Graph the first 4 (non-zero) terms on the same graph as the original function. (5 points)

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$\ln(1+x^3) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{3n}}{n}$$

$$x^2 \ln(1+x^3) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{3n+2}}{n}$$

$$P_4(x) = x^5 - \frac{x^8}{2} + \frac{x^{11}}{3} - \frac{x^{14}}{4}$$



20. Use a series to evaluate $\lim_{x\to 0} \frac{x-\ln(1+x)}{x^2}$. (4 points)

$$\lim_{x\to 0} \frac{x-\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\right)}{x^2} = \lim_{x\to 0} \frac{\frac{x^2}{2}-\frac{x^3}{3}+\frac{x^4}{4}-\cdots}{x^2}$$

$$\lim_{x\to 0} \left(\frac{1}{2} - \cancel{\frac{x}{3}} + \cancel{\frac{x^2}{4}} - \cdots\right) = \boxed{\frac{1}{2}}$$

Some useful formulas:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}, \qquad \sum_{n=1}^{\infty}1/n^4=\frac{\pi^4}{90}$$

$$f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)(x-c)^n}{n!}$$

$$|R_n(x)|\le\frac{\max|f^{n+1}(z)|}{(n+1)!}x^{n+1}$$

$$\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n=1+x+x^2+x^3+\ldots \qquad R=1$$

$$e^x=\sum_{n=0}^{\infty}\frac{x^n}{n!}=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots \qquad R=\infty$$

$$\sin x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots \qquad R=\infty$$

$$\cos x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots \qquad R=\infty$$

$$\tan^{-1}x=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\ldots \qquad R=1$$

$$\ln(1+x)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots \qquad R=1$$

$$(1+x)^k=\sum_{n=0}^{\infty}\binom{k}{n}x^n=1+kx+\frac{k(k-1)}{2!}x^2+\frac{k(k-1)(k-2)}{3!}x^3+\ldots \qquad R=1$$

$$\arcsin x=x+\frac{x^3}{2\cdot 3}+1\cdot\frac{3x^5}{2\cdot 4\cdot 5\cdot 7}+\cdots+\frac{(2n)!\,x^{2n+1}}{(2^n n!)^2(2n+1)} \qquad -1<x<1$$

$$\cosh x=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots=\sum_{n=0}^{\infty}\frac{1}{(2n)!}x^{2n} \qquad (-\infty,\infty)$$

$$\sinh x=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots=\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1} \qquad (-\infty,\infty)$$