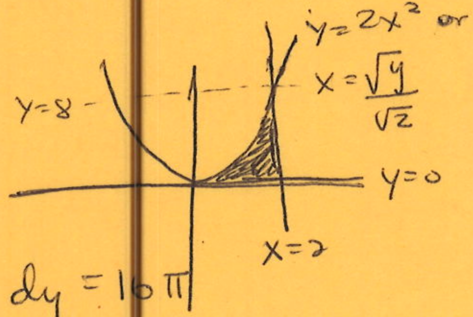
192 Homework #13 Key

1a. i. around y-axis



Shell: $2\pi\int_0^2 x(2x^2)\,dx = 16\pi$

Washer: $\pi\int_0^8 (2)^2-\left(\frac{\sqrt{y}}{\sqrt{2}}\right)^2 dy = \pi\int_0^8 4-\frac{y}{2}\,dy = 16\pi$

ii. around $y=8$

Washer: $\pi\int_0^2 (0-8)^2-(8-2x^2)^2\,dx = \pi\int_0^2 64-(64-32x^2+4x^4)\,dx$

$= \pi\int_0^2 32x^2-4x^4\,dx = \frac{896\pi}{15}$

Shell: $2\pi\int_0^8 (8-y)\left(2-\frac{\sqrt{y}}{\sqrt{2}}\right) dy = \frac{896\pi}{15}$

iii. around x-axis

Washer: $\pi\int_0^2 (2x^2)^2-(0)^2\,dx = \pi\int_0^2 4x^4\,dx = \frac{128\pi}{5}$

Shell: $2\pi\int_0^8 y\left(2-\frac{\sqrt{y}}{\sqrt{2}}\right) dy = \frac{128\pi}{5}$

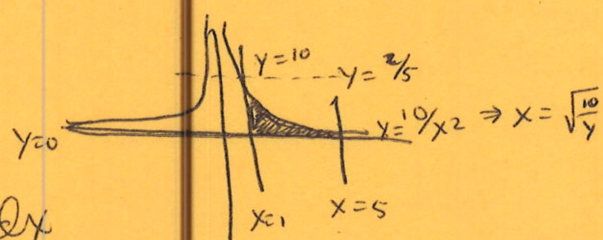
iv. around $x=2$

Shell: $2\pi\int_0^2 (2-x)(2x^2)\,dx = \frac{16\pi}{3}$

Washer: $\pi\int_0^8 \left(2-\frac{\sqrt{y}}{\sqrt{2}}\right)^2 dy = \frac{16\pi}{3}$

b. $y=\frac{10}{x^2}$, $y=0$, $x=1$, $x=5$

i. around y-axis



Shell: $2\pi\int_1^5 x\left(\frac{10}{x^2}\right) dx = 2\pi\int_1^5 \frac{10}{x}\,dx$

$= 20\pi \ln 5$

Washer: $\pi\int_0^{2/5} 5^2-1^2\,dy + \pi\int_{2/5}^{10}\left(\sqrt{\frac{10}{y}}\right)^2-1^2\,dy = \frac{48\pi}{5} + \pi\int_{2/5}^{10}\frac{10}{y}-1\,dy =$

$\cancel{\frac{48\pi}{5}} + \pi(10\ln 10 - \cancel{10} - 10\ln 2/5 + 2/5) = 10\pi(\ln 10 - \ln 2/5) =$

$20\pi \ln 5$

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1b cont'd

ii. $y=10$ (around)

Washer: $\pi \int_1^5 10^2 - \left(10 - \frac{10}{x^2}\right)^2 dx = \frac{1904\pi}{15}$

Shell: $2\pi \int_{2/5}^{10} (10-y)\left(\sqrt{\frac{10}{y}} - 1\right) dy + 2\pi \int_0^{2/5} (10-y)(5-1)\, dy$

$\frac{3584 \cdot 2\pi}{75} + \frac{392 \cdot 2\pi}{25} = \frac{1904\pi}{15}$

iii. around x-axis

Washer: $\pi \int_1^5 \left(\frac{10}{x^2}\right)^2 dx = \frac{496\pi}{15}$

Shell: $2\pi \int_0^{2/5} y(5-1)\, dy + 2\pi \int_{2/5}^{10} y\left(\sqrt{\frac{10}{y}} - 1\right) dy$

$\frac{16\pi}{25} + \frac{2432\pi}{75} = \frac{496\pi}{15}$

iv. around $x=5$

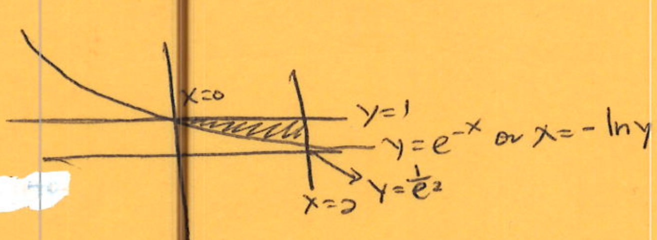
Shell: $2\pi \int_1^5 (5-x)\left(\frac{10}{x^2}\right) dx = 2\pi \int_1^5 \frac{50}{x^2} - \frac{10}{x}\, dx = 80\pi - 20\pi \ln 5$

Washer: $\pi \int_0^{2/5} (1-5)^2 - 0\, dy + \pi \int_{2/5}^{10} (1-5)^2 - \left(5 - \sqrt{\frac{10}{y}}\right)^2 dy =$

$\frac{32\pi}{5} + \pi \int_{2/5}^{10} 16 - \left(25 - 10\sqrt{\frac{10}{y}} + \frac{10}{y}\right) dy =$

$\frac{32\pi}{5} + \frac{368\pi}{5} - (10 \ln 10)\pi + 10\pi \ln \frac{2}{5} = 80\pi - 20\pi \ln 5$

c. $y = e^{-x}$, $y=1$, $x=2$



around

i. y-axis

Shell: $2\pi \int_0^2 x(1-e^{-x})\, dx = 2\pi\left(1 + \frac{3}{e^2}\right)$

Washer: $\pi \int_{1/e^2}^1 2^2 - (-\ln y)^2 dy = \pi\left(2 + \frac{6}{e^2}\right)$

ii. around x-axis

Washer: $\pi \int_0^2 1^2 - (e^{-x})^2 dx = \frac{\pi}{2}\left(3 + \frac{1}{e^4}\right)$

Shell: $2\pi \int_{1/e^2}^1 y(2 + \ln y)\, dy = \frac{\pi}{2}\left(3 + \frac{1}{e^4}\right)$

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1.c. cont'd.

iii. $y=1$ around

washer: $\pi\int_0^2 (e^{-x}-1)^2\,dx = \frac{\pi}{2}\left(1-\frac{1}{e^4}+\frac{4}{e^2}\right)$

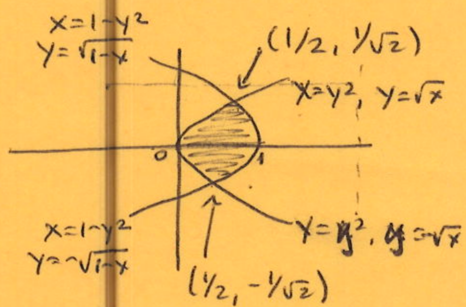
shell: $2\pi\int_{1/e^2}^{1} (1-y)(2+\ln y)\,dy = \frac{2\pi}{4}\left(1-\frac{1}{e^4}+\frac{4}{e^2}\right)$

iv. around $x=3$

Shell: $2\pi\int_0^2 (3-x)(1-e^{-x})\,dx = 4\pi$

washer $= \pi\int_{1/e^2}^{1} -(3-2)^2 + (3+\ln y)^2\,dy = 4\pi$

d. $x=y^2,\ x=1-y^2$



$y^2 = 1-y^2$ $x=1/2$

$2y^2=1$ $1-x=y^2$

$y=\pm 1/\sqrt{2}$ $y=\pm\sqrt{1-x}$

i. around $y=1$

washer: $\pi\int_0^{1/2} (1+\sqrt{x})^2-(1-\sqrt{x})^2\,dx + \pi\int_{1/2}^{1} (1+\sqrt{1-x})^2-(1-\sqrt{1-x})^2\,dx =$

$\frac{2\pi\sqrt{2}}{3} + \frac{2\pi\sqrt{2}}{3} = \frac{4\pi\sqrt{2}}{3}$

Shell: $2\pi\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (1-y)\left[(1-y^2)-y^2\right]dy = \frac{4\pi\sqrt{2}}{3}$

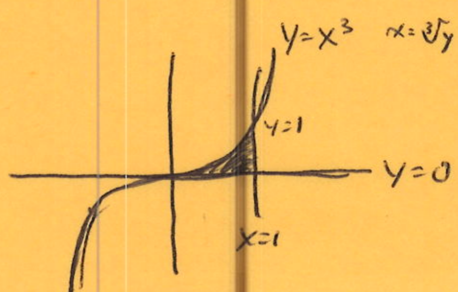
ii. around $x=3$

Shell: $2\pi\int_0^{1/2}(3-x)\sqrt{x}\,dx\ (\times 2) + 2\pi\int_{1/2}^{1}(3-x)(\sqrt{1-x})\,dx\ (\times 2) =$

$2(2\pi)\frac{5}{3\sqrt{2}} = \frac{20\pi}{3\sqrt{2}}$

washer: $\pi\int_{-1/\sqrt{2}}^{1/\sqrt{2}} (3-y^2)^2-(3-(1-y^2))^2\,dy = \frac{10\pi\sqrt{2}}{3} = \frac{20\pi}{3\sqrt{2}}$

1e. $y=x^3,\ y=0,\ x=1$



i. around y-axis

1e. cont'd.

i. Shell: $2\pi \int_0^1 x(x^3)\,dx = \frac{2\pi}{5}$

Washer: $\pi \int_0^1 1^2 - (\sqrt[3]{y})^2\,dy = \frac{2\pi}{5}$

ii. around x-axis:

Washer: $\pi \int_0^1 (x^3)^2\,dx = \frac{\pi}{7}$

Shell: $2\pi \int_0^1 y(1-\sqrt[3]{y})\,dy = \frac{\pi}{7}$

iii. around $x=1$

Shell: $2\pi \int_0^1 (1-x)x^3\,dx = \frac{\pi}{10}$

Washer: $\pi \int_0^1 (1-\sqrt[3]{y})^2\,dy = \frac{\pi}{10}$

iv. around $y=1$

Washer: $\pi \int_0^1 1^2 - (1-x^3)^2\,dx = \frac{5\pi}{14}$

Shell: $2\pi \int_0^1 (1-y)(1-\sqrt[3]{y})\,dy = \frac{5\pi}{14}$

2. $y = \frac{x^3}{6} + \frac{1}{2x}$ $[1, 2]$

$y' = \frac{x^2}{2} - \frac{1}{2x^2} = \frac{1}{2}x^2 - \frac{1}{2}x^{-2}$

around x-axis

$$2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\sqrt{1+\left(\tfrac{1}{2}x^2 - \tfrac{1}{2}x^{-2}\right)^2}\,dx = 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\sqrt{1 + \tfrac{1}{4}x^4 - \tfrac{1}{2} + \tfrac{1}{4}x^{-4}}\,dx$$

$$= 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\sqrt{\tfrac{1}{4}x^4 + \tfrac{1}{2} + \tfrac{1}{4}x^{-4}}\,dx = 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\sqrt{\left(\tfrac{1}{2}x^2 + \tfrac{1}{2}x^{-2}\right)^2}\,dx$$

$$= 2\pi \int_1^2 \left(\frac{x^3}{6} + \frac{1}{2x}\right)\left(\frac{1}{2}x^2 + \frac{1}{2x^2}\right)dx = 2\pi \int_1^2 \frac{x^5}{12} + \frac{x^3}{4} + \frac{x}{12} + \frac{1}{4x}\,dx$$

$$= 2\pi\left(\frac{1}{16}\right)(31 + \ln 16) = \frac{\pi}{8}(31 + \ln 16)$$

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2 cont'd around y-axis

$$2\pi \int_1^2 x\sqrt{1+(\tfrac{1}{2}x^2-\tfrac{1}{2}x^{-2})^2}\,dx = 2\pi\int_1^2 x\left(\tfrac{1}{2}x^2+\tfrac{1}{2}x^{-2}\right)dx = 2\pi\int_1^2 \tfrac{1}{2}x^3+\tfrac{1}{2x}\,dx$$

$$= 2\pi\left(\tfrac{1}{8}\right)(15+\ln 16) = \tfrac{\pi}{4}(15+\ln 16)$$

3a. $x=t,\ y=4-2t\quad [0,4]$

$\frac{dx}{dt}=1 \quad \frac{dy}{dt}=-2 \quad \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{1+4}=\sqrt{5}$

i. around x-axis

$2\pi\int_0^4 (4-2t)\sqrt{5}\,dt = 16\sqrt{5}\pi$

ii. around y-axis

$2\pi\int_0^4 t\sqrt{5}\,dt = 16\sqrt{5}\pi$

b. $x=t^3\quad y=t^2\quad [0,1]$

$\frac{dx}{dt}=3t^2\quad \frac{dy}{dt}=2t\quad \sqrt{\left(\frac{dy}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}=\sqrt{9t^4+4t^2}$

i. around x-axis

$2\pi\int_0^1 t^2\sqrt{9t^4+4t^2}\,dt =$

$2\pi\left[\frac{64+247\sqrt{13}}{1215}\right]$

ii. around y-axis

$2\pi\int_0^1 t^3\sqrt{9t^4+4t^2}\,dt =$

$2\pi\left[\frac{1}{243}\right]\left(42\sqrt{13}+4\sinh^{-1}\left(\frac{3}{2}\right)\right)$

c. $x=\sin t\quad y=\sin 2t\quad [0,\pi/2]$

$\frac{dx}{dt}=\cos t\quad \frac{dy}{dt}=2\cos 2t\quad \sqrt{\cos^2 t+4\cos^2 2t}=\sqrt{\cos^2 t+4(2\cos^2 t-1)^2}$

$\sin 2t = 2\sin t\cos t$

$=\sqrt{\cos^2 t+4(4\cos^4 t-4\cos^2 t+1)}=$

$\sqrt{\cos^2 t+16\cos^4 t-16\cos^2 t+4}$

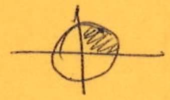
$=\sqrt{16\cos^4 t-15\cos^2 t+4}$

i around x-axis $2\pi\int_0^{\pi/2} 2\sin t\cos t\sqrt{16\cos^4 t-15\cos^2 t+4}\,dt = 2\pi(1.2778)$

ii. around y axis $2\pi\int_0^{\pi/2}\sin t\sqrt{16\cos^4 t-15\cos^2 t+4}\,dt \approx 2\pi(1.41016)$

3 cont'd

d. $x = a\cos\theta \quad y = a\sin\theta \quad [0, 2\pi]$



$\frac{dx}{d\theta} = -a\sin\theta \quad \frac{dy}{d\theta} = a\cos\theta \qquad \sqrt{a^2\sin^2\theta + a^2\cos^2\theta} = a$

i. around x-axis

$2\pi\int_0^{2\pi} a \sin\theta \cdot a\, d\theta = 2a^2\pi\int_0^{2\pi}\sin\theta\, d\theta = 2a^2\pi\cos\theta\Big|_0^{2\pi} = 0$ ✗

use symmetry $2\pi\cdot 4\int_0^{\pi/2} a^2\sin\theta\, d\theta = -8\pi a^2\cos\theta\Big|_0^{\pi/2} =$

$8a^2\pi$

ii around y-axis

$2\pi\int_0^{2\pi} a\cos\theta \cdot a\, d\theta = 2a^2\pi\int_0^{2\pi}\cos\theta\, d\theta = 2a^2\pi\sin\theta\Big|_0^{2\pi} = 0$ ✗

use symmetry $4\cdot 2\pi\int_0^{\pi/2} a^2\cos\theta\, d\theta = 8a^2\pi\sin\theta\Big|_0^{\pi/2} = 8a^2\pi$

given the symmetry of the circle, it makes sence the surface area is the same both ways

e. $x = t\cos t \quad y = t\sin t \quad [0, \pi/2]$

$\frac{dx}{dt} = \cos t - t\sin t \quad \frac{dy}{dt} = \sin t + t\cos t$

$\sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2} = \sqrt{\cos^2 t - \cancel{2t\cos t\sin t} + t^2\sin^2 t + \sin^2 t + \cancel{2t\cos t\sin t} + t^2\cos^2 t}$

$= \sqrt{1+t^2}$

i. around x-axis $2\pi\int_0^{\pi/2} t\sin t\sqrt{1+t^2}\, dt \approx 2\pi(1.53425)$

ii. around y-axis $2\pi\int_0^{\pi/2} t\cos t\sqrt{1+t^2}\, dt \approx 2\pi(0.7543)$

3 cont'd

f. $x = e^t - t,\ y = 4e^{t/2}$ $[0,1]$

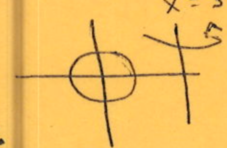
$\frac{dx}{dt} = e^t - 1 \quad \frac{dy}{dt} = 2e^{t/2} \quad \sqrt{(e^t-1)^2 + 4e^t} = \sqrt{e^{2t} - 2e^t + 1 + 4e^t}$

$= \sqrt{e^{2t} + 2e^t + 1} = \sqrt{(e^t+1)^2} = e^t + 1$

i. around x-axis $2\pi \int_0^1 4e^{t/2}(e^t+1)\,dt = \frac{16\pi}{3}(2 + (e-3)\sqrt{e})$

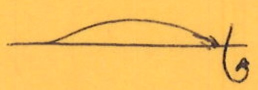
ii. around y-axis $2\pi \int_0^1 (e^t - t)(e^t+1)\,dt = \frac{\pi}{2}(e-2)e$

4. $x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4-x^2}$



$2 \cdot 2\pi \int_{-2}^{2} (3-x)\sqrt{4-x^2}\,dx = 6\pi(4\pi) = 24\pi^2$

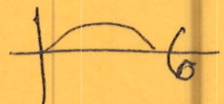
5. $y = \frac{1}{8}x^2\sqrt{2-x}$



$\pi \int_0^2 \left[\frac{1}{8}x^2\sqrt{2-x}\right]^2 dx = \frac{\pi}{64}\int_0^2 x^4(2-x)\,dx = \frac{\pi}{64}\int_0^2 2x^4 - x^5\,dx$

$= \frac{\pi}{64}\left[\frac{2}{5}x^5 - \frac{1}{6}x^6\right|_0^2 = \frac{\pi}{64}\left[\frac{2}{5}\cdot 32 - \frac{1}{6}\cdot 64\right] = \frac{\pi}{64}\cdot\frac{32}{15} = \frac{\pi}{30}$

6. $y = \frac{1}{3}x^{1/2} - x^{3/2}$ $[0, 1/3]$



$y' = \frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2}$

$\sqrt{1 + (\frac{1}{6}x^{-1/2} - \frac{3}{2}x^{1/2})^2} =$

$\sqrt{1 + \frac{1}{36x} - 2(\frac{1}{6})(\frac{3}{2})x^0 + \frac{9}{4}x} =$

$-1/2$

$\sqrt{\frac{1}{36x} + \frac{1}{2} + \frac{9}{4}x} = \sqrt{(\frac{1}{6x^{1/2}} + \frac{3}{2}x^{1/2})^2}$

$= \frac{1}{6x^{1/2}} + \frac{3}{2}x^{1/2}$

$2\pi \int_0^{1/3} (\frac{1}{3}x^{1/2} - x^{3/2})(\frac{1}{6}x^{-1/2} + \frac{3}{2}x^{1/2})\,dx$

$= 2\pi \int_0^{1/3} \frac{1}{18} + \frac{1}{2}x - \frac{1}{6}x - \frac{3}{2}x^2\,dx$

$= 2\pi \int_0^{1/2} \frac{1}{18} + \frac{1}{3}x - \frac{3}{2}x^2\,dx =$

$2\pi\left[\frac{1}{18}x + \frac{1}{6}x^2 - \frac{1}{2}x^3\right|_0^{1/3} = 2\pi\left[\frac{1}{54} + \cancel{\frac{1}{54}} - \cancel{\frac{1}{54}}\right]$

$= \pi/27$

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7. $y = e^x,\ y = e^{-x},\ x = 1$

$$2\pi\int_0^1 x(e^x - e^{-x})\,dx =$$

$$4\pi\int_0^1 x\,\frac{(e^x - e^{-x})}{2}\,dx = 4\pi\int_0^1 x\sinh x\,dx$$

+/-	u	dv
+	x	sinh x
−	1	cosh x
+	0	sinh x

$$4\pi\left[x\cosh x - \sinh x\right]_0^1 =$$

$$4\pi\left[\cosh(1) - \sinh(1) - 0\right] =$$

$$4\pi\left[\cosh(1) - \sinh(1)\right] = \cancel{4}^{2}\pi\left[\frac{e^1 + e^{-1}}{\cancel{2}} - \frac{e^1 - e^{-1}}{\cancel{2}}\right] =$$

$$2\pi\left[\cancel{e} + \frac{1}{e} - \cancel{e} + \frac{1}{e}\right] = \frac{2\pi}{e}$$