MAT 201, Exam #2, Fall 2016 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Find the a) velocity, b) acceleration and c) speed of a particle defined by the position vector $\vec{r}(t) = \tan t\,\hat{\imath} + \sec t\,\hat{\jmath} + \ln(1+3t)\hat{k}$. (7 points)

$$r'(t) = \sec^2 t\,\hat{\imath} + \sec t \tan t\,\hat{\jmath} + \frac{3}{1+3t}\hat{k}$$

$$r''(t) = 2\sec^2 t \tan t\,\hat{\imath} + (\sec t \tan^2 t + \sec^3 t)\hat{\jmath} - \frac{9}{(1+3t)^2}\hat{k}$$

$$\|r'(t)\| = \sqrt{\sec^4 t + \sec^2 t \tan^2 t + \frac{9}{(1+3t)^2}}$$

2. Evaluate $\int \left(t^2 \ln t\,\hat{\imath} + 3\sin^2 t \cos t\,\hat{\jmath} + \frac{1}{t^2}\hat{k}\right) dt$. (4 points)

$\int t^2 \ln t\,dt$ $\quad u = \ln t \quad dv = t^2$ $\quad du = \frac{1}{t} \quad v = \frac{1}{3}t^3$

$$\left(\frac{1}{3}t^3 \ln t - \int \frac{1}{3}t^2\,dt\right)\hat{\imath} + (\sin^3 t + C_2)\hat{\jmath} + \left(\frac{-1}{t} + C_3\right)\hat{k}$$

$$\left(\frac{1}{3}t^3 \ln t - \frac{1}{9}t^3 + C_1\right)\hat{\imath} + (\sin^3 t + C_2)\hat{\jmath} + \left(\frac{-1}{t} + C_3\right)\hat{k}$$

3. Find the first partial derivatives of $f(x,y) = x^y$. (4 points)

$= e^{(\ln x)y}$

$$f_x = yx^{y-1}$$

$$f_y = x^y \cdot \ln x$$

4. Find $\vec{r}_u \times \vec{r}_v$ for $\vec{r}(u,v) = 2uv\hat{\imath} + u^2\hat{\jmath} + v\hat{k}$. (6 points)

$$\vec{r}_u = 2v\hat{\imath} + 2u\hat{\jmath} + 0\hat{k}$$
$$\vec{r}_v = 2u\hat{\imath} + 0\hat{\jmath} + 1\hat{k}$$

$$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2v & 2u & 0 \\ 2u & 0 & 1 \end{vmatrix} = (2u-0)\hat{\imath} - (2v-0)\hat{\jmath} + (0-4u^2)\hat{k}$$
$$2u\hat{\imath} - 2v\hat{\jmath} - 4u^2\hat{k}$$

5. Verify that $u(x,y) = \sin x \cosh y + \cos x \sinh y$ satisfies the equation $u_{xx} + u_{yy} = 0$. (5 points)

$$u_x = \cos x \cosh y - \sin x \sinh y$$
$$u_{xx} = -\sin x \cosh y - \cos x \sinh y$$

$$u_y = \sin x \sinh y + \cos x \cosh y$$
$$u_{yy} = \sin x \cosh y + \cos x \sinh y$$

$$u_{xx} + u_{yy} = -\sin x \cosh y - \cos x \sinh y + \sin x \cosh y + \cos x \sinh y = 0 \checkmark$$

6. Find $\vec{\nabla} f$ and $\vec{\nabla}^2 f$ for $f(x,y,z) = xe^y + ye^z + ze^x$. (8 points)

$$\nabla f = \langle e^y + ze^x,\ xe^y + e^z,\ ye^z + e^x \rangle$$
$$\nabla^2 f = ze^x + xe^y + ye^z$$

$\nabla^2 f = f$!!

7. Find $\vec{\nabla} \times \vec{F}$ and $\vec{\nabla} \cdot \vec{F}$ for $\vec{F}(x,y,z) = (xy\cosh xy + \sinh xy)\hat{\imath} + x^2 \cosh xy\,\hat{\jmath} + (xy + 2z)\hat{k}$. (9 points)

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xy\cosh xy + \sinh xy & x^2\cosh xy & xy+2z \end{vmatrix} =$$

$$(x-0)\hat{\imath} - (y-0)\hat{\jmath} + (2x\cosh xy + x^2y\sinh xy - x\cosh xy - x^2y\sinh xy$$
$$- x\cosh xy)\hat{k}$$

$$= x\hat{\imath} - y\hat{\jmath} + 0\hat{k}$$

$$\vec{\nabla}\cdot\vec{F} = y\cosh xy + xy^2 \cancel{\cosh}\, xy + y\cosh xy + x^3\sinh xy + 2$$
$$= 2y\cosh xy + xy^2\sinh xy + x^3\sinh xy + 2$$

8. Evaluate $\int_C xy\,ds$ on $C: \vec{r}(t) = t^2\hat{\imath} + 2t\hat{\jmath}$ on $[0,1]$. (6 points)

$$r'(t) = 2t\hat{\imath} + 2\hat{\jmath} \quad \|r'(t)\| = \sqrt{4t^2+4} = 2\sqrt{t^2+1}$$

$$\int_0^1 t^2 \cdot 2t \cdot 2\sqrt{t^2+1}\,dt = 4\int t^2 \cdot t\sqrt{t^2+1}\,dt \qquad \begin{array}{ll} u = t^2 & dv = t(t^2+1)^{1/2} \\ du = 2t & v = \frac{1}{2}(t^2+1)^{3/2}\cdot\frac{2}{3} \end{array}$$

$$= 4\left[\tfrac{1}{3}t^2(t^2+1)^{3/2} - \tfrac{2}{3}\int t(t^2+1)^{3/2}dt\right] = 4\left[\tfrac{1}{3}t^2(t^2+1)^{3/2} - \tfrac{2}{3}\cdot\tfrac{2}{5}\cdot\tfrac{1}{2}(t^2+1)^{5/2}\right]_0^1$$

$$= \tfrac{4}{3}t^2(t^2+1)^{3/2} - \tfrac{8}{15}(t^2+1)^{5/2}\Big|_0^1 = \tfrac{4}{3}(2)^{3/2} - \tfrac{8}{15}(2)^{5/2} + \tfrac{8}{15}(1)^{5/2}$$

$$\approx 1.28758\ldots$$

9. Evaluate $\int_C z^2dx + x^2dy + y^2dz$ on the line segment from $(1,0,0)$ to $(4,1,2)$. (8 points)

$$\vec{r}(t) = (3t+1)\hat{\imath} + t\hat{\jmath} + 2t\hat{k}$$
$$r'(t) = \langle 3, 1, 2\rangle$$

$$\int_0^1 \left((2t)^2\cdot 3 + (3t+1)^2\cdot 1 + t^2\cdot 2\right)dt =$$

$$\int_0^1 12t^2 + 9t^2 + 6t + 1 + 2t^2\,dt = \int_0^1 23t^2 + 6t + 1\,dt =$$

$$\tfrac{23}{3}t^3 + 3t^2 + t\Big|_0^1 = \tfrac{23}{3} + 3 + 1 = \tfrac{35}{3}$$

10. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ for $\vec{F}(x,y,z) = yz\hat{\imath} + xz\hat{\jmath} + (xy+2z)\hat{k}$ on any path between $(1,0,-2)$ and $(4,6,3)$. Use the Fundamental Theorem of Line Integrals after establishing that the field is conservative. (10 points)

$$\nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & xz & xy+2z \end{vmatrix} = (x-x)\hat{\imath} - (y-y)\hat{\jmath} + (z-z)\hat{k} = \vec{0} \quad \text{Conservative}$$

$$\int yz\,dx = xyz + a(y,z) \qquad f(x,y,z) = xyz + z^2 + K$$

$$\int xz\,dy = xyz + b(x,z)$$

$$\int xy + 2z\,dz = xyz + z^2 + c(x,y)$$

$$\int_C \vec{F}\cdot d\vec{r} = f(4,6,3) - f(1,0,-2) =$$

$$4\cdot 6\cdot 3 + 3^2 - \left(1\cdot 0\cdot(-2) + 4\right) =$$

$$72 + 9 - 0 - 4 = \boxed{77}$$

11. Use Green's Theorem to evaluate $\int_C y^3dx - x^3dy$ where C is the circle $x^2 + y^2 = 4$. (7 points)

$$\frac{\partial M}{\partial y} = 3y^2 \qquad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -3x^2 - 3y^2$$

$$\frac{\partial N}{\partial x} = -3x^2$$

$$\int_0^{2\pi}\int_0^2 -3r^2\cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 -3r^3\,dr\,d\theta = \int_0^{2\pi} -\frac{3}{4}r^4\Big|_0^2\,d\theta =$$

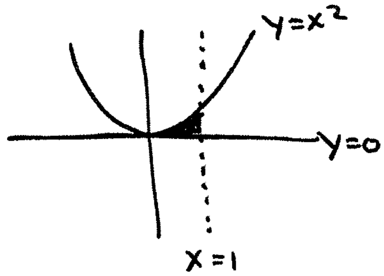
$$-12\int_0^{2\pi} d\theta = \boxed{-24\pi}$$

12. Evaluate $\int\int_R \frac{xy^2}{x^2+1} dA$ on the region $[0,1] \times [-3,3]$. (5 points)

$$\int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1} dy\, dx = \int_0^1 \frac{x}{x^2+1} \cdot \frac{1}{3}y^3 \Big|_{-3}^{3} dx = 18\int_0^1 \frac{x}{x^2+1} dx =$$

$$9 \ln|x^2+1| \Big|_0^1 = 9[\ln 2 - \ln 1] = \boxed{9 \ln 2}$$

13. Sketch the region defined by $0 \le x \le 1, 0 \le y \le x^2$. Use the sketch to set up and evaluate $\int\int_R \frac{y}{x^5+1} dA$ over that region. (7 points)

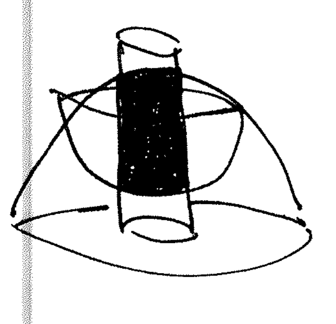


$$\int_0^1 \int_0^{x^2} \frac{y}{x^5+1} dy\, dx =$$

$$\frac{1}{2}\int_0^1 \frac{y^2}{x^5+1} \Big|_0^{x^2} dx = \frac{1}{2}\int_0^1 \frac{x^4}{x^5+1} dx =$$

$$\frac{1}{10} \ln|x^5+1| \Big|_0^1 = \frac{1}{10}[\ln 2 - \ln 1] = \boxed{\frac{1}{10}\ln 2}$$

14. Set up a double or triple integral to find the volume bounded by $z = 2x^2 + y^2, z = 8 - x^2 - 2y^2$ inside the cylinder $x^2 + y^2 = 1$. [You do not need to evaluate the integral.] (6 points)



$$2x^2 + y^2 = 8 - x^2 - 2y^2$$
$$+x^2 + 2y^2 \qquad +x^2 + 2y^2$$

$$3x^2 + 3y^2 = 8$$
$$x^2 + y^2 = \frac{8}{3}$$

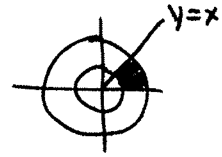
$$z = x^2 + x^2 + y^2 = r^2\cos^2\theta + r^2$$
$$z = 8 - x^2 - y^2 - y^2 = 8 - r^2 - r^2\sin^2\theta$$

$$\int_0^{2\pi}\int_0^1\int_{r^2\cos^2\theta + r^2}^{8-r^2-r^2\sin^2\theta} r\, dz\, dr\, d\theta \text{ or } \int_0^{2\pi}\int_0^1 (8 - 2r^2 - r^2\sin^2\theta - r^2\cos^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 (8-3r^2)\, r\, dr\, d\theta$$

15. Set up a double integral to evaluate $\int\int_R \arctan\left(\frac{y}{x}\right) dA$ over the region $1 \le x^2 + y^2 \le 4, 0 \le y \le x$ in polar coordinates. Sketch the region. (8 points)

$$\int_0^{\pi/4} \int_1^2 \theta r \, dr \, d\theta$$



$\arctan(y/x) = \theta$

16. Evaluate $\int_0^{\sqrt{\pi}} \int_0^x \int_0^{xz} x^2 \sin y \, dydzdx$. (8 points)

$$\int_0^{\sqrt{\pi}} \int_0^x x^2(-\cos y)\Big|_0^{xz} dz\,dx = -\int_0^{\sqrt{\pi}} \int_0^x x^2(\cos(xz) - 1)\, dz\,dx$$

$$\int_0^{\sqrt{\pi}} -\frac{x^2 \sin(xz)}{x} + z \Big|_0^x dx = \int_0^{\sqrt{\pi}} x^3 - x\sin(x^2)\, dx =$$

$$\frac{1}{4}x^4 + \frac{1}{2}\cos(x^2)\Big|_0^{\sqrt{\pi}} = \frac{1}{4}\pi^2 + \frac{1}{2}(-1) - 0 - \frac{1}{2}(1) =$$

$$\boxed{\left(\frac{\pi}{2}\right)^2 - 1}$$

17. Rewrite the integral $\int_{-3}^{3} \int_0^{\sqrt{4-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2+y^2}\, dzdydx$ as a triple integral in cylindrical coordinates. [You do not need to evaluate it.] (7 points)

$$\int_0^{\pi} \int_0^{3} \int_0^{9-r^2} r^2 \, dz\, dr\, d\theta$$

18. Rewrite the integral $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} xy\,dz\,dy\,dx$ in spherical coordinates. [You do not need to evaluate it.] (7 points)

$$\int_0^{\pi/4} \int_0^{\pi/2} \int_0^{\sqrt{2}} \rho^2 \sin^2\varphi \cos\theta \sin\theta \cdot \rho^2 \sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$= \int_0^{\pi/4} \int_0^{\pi/2} \int_0^{\sqrt{2}} \rho^4 \sin^3\varphi \cos\theta \sin\theta \, d\rho\, d\theta\, d\varphi$$

$z = \sqrt{2-x^2-y^2} \Rightarrow$

$x^2+y^2+z^2 = 2$

$\rho = \sqrt{2}$

$z = r$ $\quad \tan\varphi = 1$

$\Rightarrow \varphi = \pi/4$

19. Find the potential function, if it exists, for the vector field $\vec{F}(x,y) = 2y^{3/2}\hat{\imath} + 3x\sqrt{y}\hat{\jmath}$. If it does not exist, explain why not. (5 points)

$\vec{\nabla} \times \vec{F} = 3\sqrt{y} - 3\sqrt{y} = 0$

$$\int 2y^{3/2}\,dx = 2xy^{3/2} + g(y)$$

$$\int 3xy^{1/2}\,dy = 3x \cdot \tfrac{2}{3} y^{3/2} + h(x) = 2xy^{3/2} + h(x)$$

$$f(x,y) = 2xy^{3/2} + K$$

20. Set up a double integral to find the area of one loop of the rose $r = \cos 2\theta$. [You do not need to evaluate it.] (5 points)

$0 = \cos 2\theta$

$2\theta = \pm \pi/2 \quad \Rightarrow \theta = \pm \pi/4$

$$\int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r\,dr\,d\theta$$

or $2\int_0^{\pi/4} \int_0^{\cos 2\theta} r\,dr\,d\theta$ using symmetry