MAT 202, Exam #3, Part I, Fall 2016 Name KEY

Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. When you are finished with this portion of exam, get Part II.

1. Determine if each statement is True or False. (1 point each)

a. T F — The dot product of two vectors $\vec{u}$ and $\vec{v}$ is another vector represented by
$$\vec{u}\cdot\vec{v} = \begin{bmatrix} u_1v_1 \\ u_2v_2 \\ \vdots \\ u_nv_n \end{bmatrix}.$$

b. T F — If $\vec{u}\cdot\vec{v} < 0$, then the angle θ between $\vec{u}$ and $\vec{v}$ is obtuse.

c. T F — A nonzero vector in an inner product space can have a norm of zero.

d. T F — The norm of the vector $\vec{u}$ is defined as the angle between the vector and the positive x-axis.

e. T F — An orthonormal basis derived by the Gram-Schmidt orthonormalization process does not depend on the order of the vectors in the basis.

f. T F — A set S of vectors in an inner product space V is orthonormal when every vector is a unit vector and each pair of vectors is orthogonal.

g. T F — The set of all vectors orthogonal to every vector in a subspace S is called the orthogonal complement of S and is designated by $S^{\perp}$.

h. T F — For polynomials, the differential operator D_x is a linear transformation from $P_n \rightarrow P_{n-1}$.

i. T F — The vector spaces R^4 and P_4 are isomorphic to each other. $\mathbb{R}^4 \sim P_3$

j. T F — Any linear function of the form $f(x) = ax + b$ is a linear transformation from $R \rightarrow R$. will fail linear tests if $b \neq 0$

k. T F — The nullity is the number of free variables in a matrix.

l. T F — The range of a linear transformation from a vector space V into a vector space W is a subspace of V. Subspace of W

m. T F — The matrix of a linear transformation is defined by the effects of the transformation on the basis vectors of the space.

n. T F — The matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is onto.

o. T F The matrix $A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$ scales a vector vertically. Shear mating

2. Given the vector $\vec{u} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$, find the following: (2 points each)

a. $\|v\|$

$\sqrt{1+4+1+0} = \sqrt{6}$

b. A unit vector in the direction of v

$\begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \\ 0 \end{bmatrix}$

c. $\|\vec{u} - \vec{v}\|$

$\vec{u} - \vec{v} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} -1 \\ -2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}$ $\|\vec{u} - \vec{v}\| = \sqrt{1+4+0+9} = \sqrt{14}$

d. $\vec{u} \cdot \vec{v}$

$2+0+1+0 = 3$

e. Are $\vec{u}$ and $\vec{v}$ orthogonal? If not, is the angle between the vectors acute or obtuse?

no. The angle is acute since $\vec{u} \cdot \vec{v} > 0$.

3. Prove that the rotation matrix $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ has orthonormal columns, and $A^{-1} = A^T$. (4 points)

$A^{-1} = \frac{1}{\det A}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $\det A = \sin^2\theta + \cos^2\theta = 1$

$A^T = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ $A^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

these are equal

further $C_1 \cdot C_2 = \cos\theta\sin\theta - \cos\theta\sin\theta = 0$ & $\|C_1\| = \|C_2\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

4. Given the inner product $\langle f, g\rangle = \int_{-1}^{2} f(x)g(x)dx$, find $\|f\|$ and $d(f,g) = \|f - g\|$ for $f(x) = x + 2, g(x) = x^2 - 1$. (5 points)

$\sqrt{\int_{-1}^{2}(x+2)^2 dx} = \sqrt{\int_{-1}^{2} x^2 + 4x + 4\, dx} = \sqrt{\frac{1}{3}x^3 + 2x^2 + 4x \Big|_{-1}^{2}} = \sqrt{21} = \|f\|$

$f - g = x + 2 - (x^2 - 1) = -x^2 + x + 3$ $(-x^2 + x + 3)^2 = x^4 - 2x^3 - 6x^2 + x^2 + 6x + 9$

$\sqrt{\int_{-1}^{2}(-x^2 + x + 3)^2 dx} = \sqrt{\int_{-1}^{2} x^4 - 2x^3 - 5x^2 + 6x + 9\, dx} = \sqrt{\frac{x^5}{5} - \frac{1}{2}x^4 - \frac{5}{3}x^3 + 3x^2 + 9x \Big]_{-1}^{2}}$

$= \sqrt{\frac{201}{10}} = \|f - g\|$

5. Determine if $T: R^2 \to R^2, T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 4x_2 \\ 3x_2 \end{bmatrix}$ is a linear transformation. Prove it if it is; find a counterexample if it is not, and state the property that is violated. (5 points)

$T = A = \begin{bmatrix} 1 & -4 \\ 0 & 3 \end{bmatrix}$ it is linear.

$T(\vec{0}) = \begin{bmatrix} 0 - 0 \\ 0 \end{bmatrix}$ ✓

$T\begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix} = \begin{bmatrix} x_1 + y_1 - 4x_1 - 4x_2 \\ 3x_2 + 3x_2 \end{bmatrix}$ $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) + T\left(\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 4x_2 \\ 3x_2 \end{bmatrix} + \begin{bmatrix} y_1 - 4y_2 \\ 3y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_1 - 4x_2 - 4y_2 \\ 3x_2 + 3y_2 \end{bmatrix}$ ✓

$T\left(\begin{bmatrix} kx_1 \\ kx_2 \end{bmatrix}\right) = \begin{bmatrix} kx_1 - 4kx_2 \\ 3kx_2 \end{bmatrix}$ $kT\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = k\begin{bmatrix} x_1 - 4x_2 \\ 3x_2 \end{bmatrix} = \begin{bmatrix} kx_1 - 4kx_2 \\ 3kx_2 \end{bmatrix}$ ✓

6. Use Gram-Schmidt to find an orthogonal basis for the space spanned by $\left\{\begin{bmatrix}0\\2\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\3\end{bmatrix}\right\}$. (7 points)

$\vec{v_1} = \begin{bmatrix}0\\2\\1\end{bmatrix}$

$\vec{v_2} = \begin{bmatrix}1\\-1\\1\end{bmatrix} - \left(\frac{0-2+1}{0+4+1}\right)\begin{bmatrix}0\\2\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix} + \frac{1}{5}\begin{bmatrix}0\\2\\1\end{bmatrix} = \begin{bmatrix}1\\-1\\1\end{bmatrix} + \begin{bmatrix}0\\2/5\\1/5\end{bmatrix} = \begin{bmatrix}1\\-3/5\\6/5\end{bmatrix}$

$\vec{v_2} = \begin{bmatrix}5\\-3\\6\end{bmatrix}$

$\vec{v_3} = \begin{bmatrix}1\\1\\3\end{bmatrix} - \left(\frac{0+2+3}{0+4+1}\right)\begin{bmatrix}0\\2\\1\end{bmatrix} - \left(\frac{5-3+18}{25+9+36}\right)\begin{bmatrix}5\\-3\\6\end{bmatrix} = \begin{bmatrix}1\\1\\3\end{bmatrix} - \begin{bmatrix}0\\2\\1\end{bmatrix} - \frac{2}{7}\begin{bmatrix}5\\-3\\6\end{bmatrix} =$

$\begin{bmatrix}1\\1\\3\end{bmatrix} + \begin{bmatrix}0\\-2\\-1\end{bmatrix} + \begin{bmatrix}-10/7\\6/7\\-12/7\end{bmatrix} = \begin{bmatrix}-3/7\\-1/7\\2/7\end{bmatrix}$

$\vec{v_3} = \begin{bmatrix}-3\\-1\\2\end{bmatrix}$

$\left\{\begin{bmatrix}0\\2\\1\end{bmatrix}, \begin{bmatrix}5\\-3\\6\end{bmatrix}, \begin{bmatrix}-3\\-1\\2\end{bmatrix}\right\}$

7. For $W = span\left\{\begin{bmatrix}1\\2\\-1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\\-3\end{bmatrix}\right\}$, find the projection of $\vec{y} = \begin{bmatrix}4\\1\\5\\1\end{bmatrix}$ onto W, and its orthogonal complement in $W^{\perp}$. (5 points)

$\vec{y_{\parallel}} = \left(\frac{4+2-5+1}{1+4+1+1}\right)\begin{bmatrix}1\\2\\-1\\1\end{bmatrix} + \left(\frac{-4+1+0-3}{1+1+0+9}\right)\begin{bmatrix}-1\\1\\0\\-3\end{bmatrix} =$

$\frac{2}{7}\begin{bmatrix}1\\2\\-1\\1\end{bmatrix} + \frac{-6}{11}\begin{bmatrix}-1\\1\\0\\-3\end{bmatrix} = \begin{bmatrix}2/7\\4/7\\-2/7\\2/7\end{bmatrix} + \begin{bmatrix}6/11\\-6/11\\0\\18/11\end{bmatrix} = \begin{bmatrix}64/77\\2/77\\-2/7\\148/77\end{bmatrix}$

$\vec{y_{\perp}} = \begin{bmatrix}4\\1\\5\\1\end{bmatrix} - \begin{bmatrix}64/77\\2/77\\-2/7\\148/77\end{bmatrix} = \begin{bmatrix}244/77\\75/77\\37/7\\-71/77\end{bmatrix}$

8. Write the matrix for the linear transformation for the linear transformation given by $T(x^k) = \int_0^x t^k\,dt$ on the standard basis for $P_4 \to P_5$. Is the transformation onto? (5 points)

P_4 basis $\{1, x, x^2, x^3, x^4\}$

$T(1) = \int_0^x 1\,dt = x$

$T(x) = \int_0^x t\,dt = \frac{1}{2}x^2$

$T(x^2) = \int_0^x t^2\,dt = \frac{1}{3}x^3$

$T(x^3) = \int_0^x t^3\,dt = \frac{1}{4}x^4$

$T(x^4) = \int_0^x t^4\,dt = \frac{1}{5}x^5$

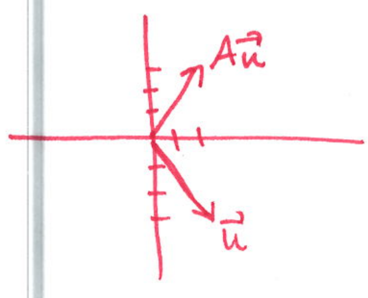
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 1/5 \end{bmatrix}$$

not onto

9. Given the vector $\vec{u} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and the transformations described below, write the matrix of the transformation and apply it to $\vec{u}$ and graph both $\vec{u}$ and $A\vec{u}$. (2 points each)

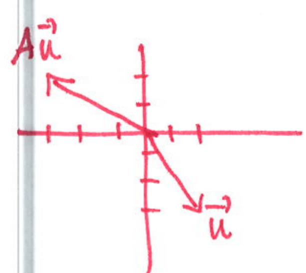
a. Reflection over the x-axis

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$



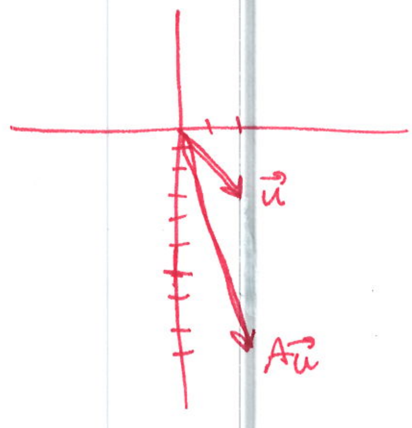
b. Reflection over the line $y = x$

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$



c. Vertical stretch by a factor of 3

$A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -9 \end{bmatrix}$



MAT 202, Exam #3, Part II, Fall 2016 Name KEY

Instructions: Show all work. You **may** use a calculator on this portion of the exam. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question. Provide explanations where requested.

1. Given the linear transformation defined by $A = \begin{bmatrix} 3 & -2 & 6 & -1 & 15 \\ 4 & 3 & -8 & 10 & -14 \\ 2 & -3 & 4 & -4 & 20 \\ 0 & 6 & 1 & 2 & 8 \end{bmatrix}$, find a basis for the kernel and range of the transformation. (6 points)

rref $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 27/5 \\ 0 & 1 & 0 & 0 & 14/5 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -22/5 \end{bmatrix}$

$x_1 = -27/5\, x_5$
$x_2 = -14/5\, x_5$
$x_3 = 0$
$x_4 = 22/5\, x_5$
$x_5 = x_5$

$\ker A = \left\{ \begin{bmatrix} -27 \\ -14 \\ 0 \\ 22 \\ 5 \end{bmatrix} \right\}$

range = all of $\mathbb{R}^4$

basis $\left\{ \begin{bmatrix} 3 \\ 4 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ -3 \\ 6 \end{bmatrix}, \begin{bmatrix} 6 \\ -8 \\ 4 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 10 \\ -4 \\ 2 \end{bmatrix} \right\}$

2. Given the linear transformation defined by $A = \begin{bmatrix} -1 & 3 & 2 & 1 & 4 \\ 2 & 3 & 5 & 0 & 0 \\ 2 & 1 & 2 & 1 & 0 \end{bmatrix}$, determine if the transformation is any of the following. Explain your reasoning in each case. (2 points each)

rref $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 10/9 & 4/9 \\ 0 & 1 & 0 & 5/3 & 8/3 \\ 0 & 0 & 1 & -13/9 & -16/9 \end{bmatrix}$

a. One-to-one

no, there is not a pivot in every column

b. Onto

yes. there is a pivot in every row

3. Given $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 \\ x_3 - x_2 \\ x_1 + 2x_2 - x_4 \\ 3x_3 + x_4 \\ 0 \end{bmatrix}$. Write the matrix of the transformation. Explain why this proves the transformation is linear. If T^{-1} exists, find its matrix. If it is not invertible, explain why not. (6 points)

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 2 & 0 & -1 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

T^{-1} does not exist since A is not invertible

this is linear since matrices satisfy linear requirements

4. Let A be an $n \times n$ matrix such that $A^2 = [\vec{0}]$. Prove that if A is similar to B then $B^2 = [\vec{0}]$. (4 points)

A similar to B means $A = PBP^{-1}$ or $P^{-1}AP = B$

$B^2 = (P^{-1}AP)^2 = P^{-1}A\cancel{PP^{-1}}AP = P^{-1}A^2P$

since $A^2 = 0$ matrix $\quad B^2 = P^{-1}(\vec{0})P = 0$ matrix

5. Consider $S = span\left\{\begin{bmatrix}1\\2\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\3\end{bmatrix}\right\}$. Find an orthogonal basis for $S^{\perp}$. (5 points)

$\begin{bmatrix}a\\b\\c\\d\end{bmatrix}$ in $S^{\perp}$

$a + 2b + c = 0$
$-2a + b + 3d = 0$

$$\begin{bmatrix}1 & 2 & 1 & 0\\-2 & 1 & 0 & 3\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & 1/5 & -6/5\\0 & 1 & 2/5 & 3/5\end{bmatrix}$$

$x_1 = -1/5 x_3 + 6/5 x_4$
$x_2 = -2/5 x_3 - 3/5 x_4$
$x_3 = x_3$
$x_4 = x_4$

$$\begin{bmatrix}-1\\-2\\5\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & 1 & 0\\-2 & 1 & 0 & 3\\-1 & -2 & 5 & 0\end{bmatrix} \xrightarrow{rref} \begin{bmatrix}1 & 0 & 0 & -6/5\\0 & 1 & 0 & 3/5\\0 & 0 & 1 & 0\end{bmatrix} \Rightarrow \begin{bmatrix}6\\-3\\0\\5\end{bmatrix}$$

$$S^{\perp} = \left\{\begin{bmatrix}-1\\-2\\5\\0\end{bmatrix}, \begin{bmatrix}6\\-3\\0\\5\end{bmatrix}\right\}$$

6. Find the QR factorization for $A = \begin{bmatrix}1 & 4\\3 & 0\\1 & 1\\1 & 2\end{bmatrix}$. (5 points)

$$\vec{V_2} = \begin{bmatrix}4\\0\\1\\2\end{bmatrix} - \left(\frac{4+0+1+2}{1+9+1+1}\right)\begin{bmatrix}1\\3\\1\\1\end{bmatrix} = \begin{bmatrix}4\\0\\1\\2\end{bmatrix} - \frac{7}{12}\begin{bmatrix}1\\3\\1\\1\end{bmatrix} = \begin{bmatrix}4\\0\\1\\2\end{bmatrix} + \begin{bmatrix}-7/12\\-21/12\\-7/12\\-7/12\end{bmatrix} = \begin{bmatrix}41/12\\-21/12\\5/12\\17/12\end{bmatrix}$$

$$Q = \begin{bmatrix}1 & 41\\3 & -21\\1 & 5\\1 & 17\end{bmatrix}$$

$$R = Q^TA = \begin{bmatrix}1 & 3 & 1 & 1\\41 & -21 & 5 & 17\end{bmatrix}\begin{bmatrix}1 & 4\\3 & 0\\1 & 1\\1 & 2\end{bmatrix} = \begin{bmatrix}1+9+1+1 & 4+0+1+2\\41-63+5+17 & 164+0+5+34\end{bmatrix}$$

$$= \begin{bmatrix}12 & 7\\0 & 203\end{bmatrix} = R$$

7. Consider a generic 8×6 matrix. Is it possible for the linear transformation defined by the matrix to be:

   a. One-to-one? (2 points)

   yes, there can be a pivot in every column

   b. Onto? (2 points)

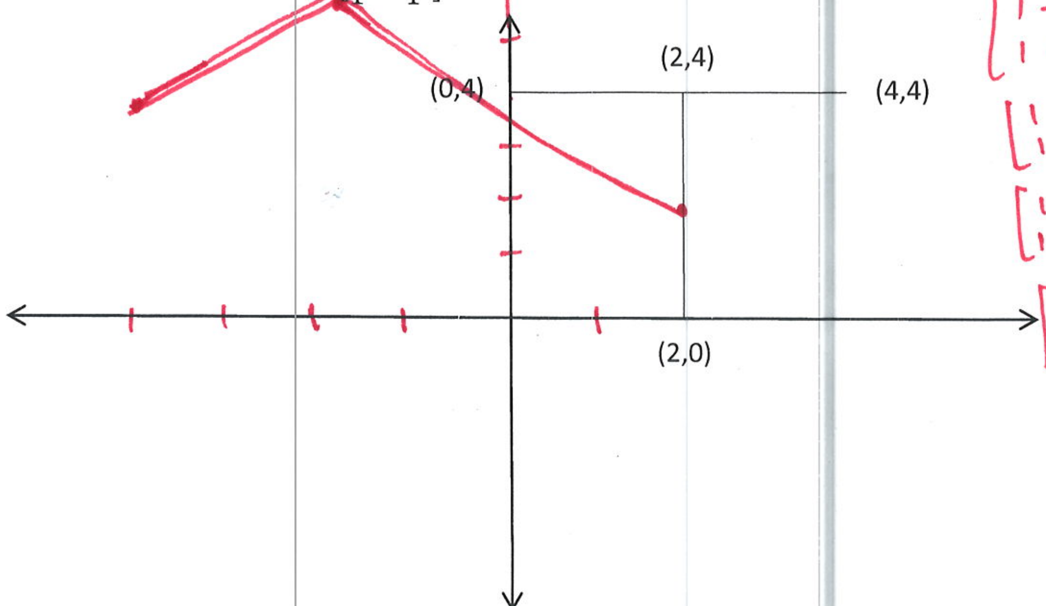
   no, there cannot be a pivot in every row (max 6 pivots)

   c. If the matrix has 6 pivots, what is the dimension of the kernel and the range? (4 points)

   $\dim(\ker A) = 0$

   $\dim(\text{range}) = 6$

8. Consider the shape T defined by the vertices shown on the graph. Apply the transformation $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and draw the resulting figure. (4 points)



$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 4 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \end{bmatrix}$

$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$

9. Given an example of an isomorphism from $R^4 \to R^4$. (2 points)

any matrix which reduces to the identity

answers will vary

10. Draw a picture to illustrate how a linear transformation maps the domain into the codomain. Be sure to include in your drawing the kernel, the $\vec{0}$ in both sets, and the range. (4 points