

1a. $\begin{bmatrix} 3 & 6 & | & -3 \\ 5 & 7 & | & 10 \end{bmatrix} \xrightarrow{\frac{1}{3}R_1 \rightarrow R_1} \begin{bmatrix} 1 & 2 & | & -1 \\ 5 & 7 & | & 10 \end{bmatrix} \xrightarrow{-5R_1 + R_2 \rightarrow R_2}$

$\begin{bmatrix} 1 & 2 & | & -1 \\ 0 & -3 & | & 15 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2 \rightarrow R_2} \begin{bmatrix} 1 & 2 & | & -1 \\ 0 & 1 & | & -5 \end{bmatrix} \xrightarrow{-2R_2 + R_1 \rightarrow R_1}$

$\begin{bmatrix} 1 & 0 & | & 9 \\ 0 & 1 & | & -5 \end{bmatrix} \rightarrow X = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$ Consistent independent

b. $\begin{bmatrix} 2 & 0 & 0 & -4 & | & -10 \\ 0 & 3 & 3 & 0 & | & 0 \\ 0 & 0 & 1 & 4 & | & -1 \\ -3 & 2 & 3 & 1 & | & 5 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} -3 \\ 5 \\ -5 \\ 1 \end{bmatrix}$ Consistent independent

c. $\begin{bmatrix} 1 & -2 & -3 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 2 & -4 & 9 & | & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ Consistent independent

d. $\begin{bmatrix} 2 & 0 & -6 & | & -8 \\ 0 & 1 & 2 & | & 3 \\ 3 & 6 & -2 & | & -4 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$ Consistent independent

e. $\begin{bmatrix} 2 & -5 & 8 & | & 0 \\ -3 & -4 & 2 & | & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 22/23 & | & 0 \\ 0 & 1 & -28/23 & | & 0 \end{bmatrix}$ Consistent dependent

$X_1 + 22/23 X_3 = 0$
 $X_2 - 28/23 X_3 = 0 \rightarrow X_1 = -22/23 X_3$
 $X_2 = 28/23 X_3$
 $X_3 = X_3$
 $X = \begin{bmatrix} -22 \\ 28 \\ 23 \end{bmatrix} t$

f. $\begin{bmatrix} 2 & 0 & 0 & -4 & 1 & | & 0 \\ 0 & 3 & 3 & 0 & -1 & | & 0 \\ 0 & 0 & 1 & 4 & 6 & | & 0 \\ -3 & 2 & 3 & 1 & -2 & | & 0 \end{bmatrix} \xrightarrow{rref} \begin{bmatrix} 1 & 0 & 0 & 0 & 97/54 & | & 0 \\ 0 & 1 & 0 & 0 & -101/27 & | & 0 \\ 0 & 0 & 1 & 0 & 92/27 & | & 0 \\ 0 & 0 & 0 & 1 & 35/54 & | & 0 \end{bmatrix}$

$X_1 + 97/54 X_5 = 0$
 $X_2 - 101/27 X_5 = 0$
 $X_3 + 92/27 X_5 = 0$
 $X_4 + 35/54 X_5 = 0$
 $X_5 = X_5$
 $X_1 = -97/54 X_5$
 $X_2 = 101/27 X_5$
 $X_3 = -92/27 X_5$
 $X_4 = -35/54 X_5$
 $X_5 = X_5$
 $X = \begin{bmatrix} -97 \\ 202 \\ -184 \\ -35 \\ 54 \end{bmatrix} t$

Consistent dependent

2a. $\begin{cases} 12a - 12b = 7 \\ 3a + 4b = 0 \end{cases} \quad \left[\begin{array}{cc|c} 12 & -12 & 7 \\ 3 & 4 & 0 \end{array} \right] \quad -4R_2 + R_1 \rightarrow R_2$

$\left[\begin{array}{cc|c} 12 & -12 & 7 \\ 0 & -28 & 7 \end{array} \right] \cdot \frac{-1}{28}R_2 \rightarrow R_2 \quad \left[\begin{array}{cc|c} 12 & -12 & 7 \\ 0 & 1 & -1/4 \end{array} \right] \quad \frac{1}{12}R_1 \rightarrow R_1$

$\left[\begin{array}{cc|c} 1 & -1 & 7/12 \\ 0 & 1 & -1/4 \end{array} \right] \quad R_1 + R_2 \rightarrow R_1 \quad \left[\begin{array}{cc|c} 1 & 0 & 1/3 \\ 0 & 1 & -1/4 \end{array} \right] \quad \begin{matrix} a = 1/3 \rightarrow x = 3 \\ b = -1/4 \rightarrow y = -4 \end{matrix}$

b. $\begin{cases} 2a + b - 2c = 5 \\ 3a - 4b = -1 \\ 2a + b + 3c = 0 \end{cases} \quad \left[\begin{array}{ccc|c} 2 & 1 & -2 & 5 \\ 3 & -4 & 0 & -1 \\ 2 & 1 & 3 & 0 \end{array} \right] \rightarrow \text{rref} \rightarrow \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\begin{matrix} a = 1 & \rightarrow & x = 1 \\ b = 1 & \rightarrow & y = 1 \\ c = -1 & \rightarrow & z = -1 \end{matrix}$

3a. $\begin{cases} a(1)^3 + b(1)^2 + c(1) + d = 7 \\ a(2)^3 + b(2)^2 + c(2) + d = 17 \\ a(3)^3 + b(3)^2 + c(3) + d = 31 \\ a(4)^3 + b(4)^2 + c(4) + d = 65 \end{cases} \quad \begin{matrix} a + b + c + d = 7 \\ 8a + 4b + 2c + d = 17 \\ 27a + 9b + 3c + d = 31 \\ 64a + 16b + 4c + d = 65 \end{matrix}$

$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 7 \\ 8 & 4 & 2 & 1 & 17 \\ 27 & 9 & 3 & 1 & 31 \\ 64 & 16 & 4 & 1 & 65 \end{array} \right] \rightarrow \text{rref} \quad \begin{bmatrix} 8/3 \\ -14 \\ 100/3 \\ -15 \end{bmatrix}$

$8/3 x^3 - 14x^2 + 100/3 x - 15 = y$

b. $\begin{cases} a(-2)^4 + b(-2)^3 + c(-2)^2 + d(-2) + e = 28 \\ a(-1)^4 + b(-1)^3 + c(-1)^2 + d(-1) + e = 0 \\ a(0)^4 + b(0)^3 + c(0)^2 + d(0) + e = -6 \\ a(1)^4 + b(1)^3 + c(1)^2 + d(1) + e = -8 \\ a(2)^4 + b(2)^3 + c(2)^2 + d(2) + e = 0 \end{cases}$

$\begin{matrix} 16a - 8b + 4c - 2d + e = 28 \\ a - b + c - d + e = 0 \\ e = -6 \end{matrix}$

$\begin{matrix} a + b + c + d + e = -8 \\ 16a + 8b + 4c + 2d + e = 0 \end{matrix}$

$\left[\begin{array}{cccc|c} 16 & -8 & 4 & -2 & 28 \\ 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & -6 \\ 1 & 1 & 1 & 1 & -8 \\ 16 & 8 & 4 & 2 & 0 \end{array} \right] \rightarrow \text{rref} \rightarrow \begin{bmatrix} 1 \\ -1 \\ 1 \\ -3 \\ -6 \end{bmatrix}$

$x^4 - x^3 + x^2 - 3x - 6 = y$

4a. $E + D = 600 + A$
 $C + 300 + D = 200$
 $300 + A = 700 + B$
 $500 + B = 100 + C$

$-A + D + E = 600$
 $C - D = -100$
 $A - B = 400$
 $B - C = -400$

$$\begin{bmatrix} -1 & 0 & 0 & 1 & 1 & 600 \\ 0 & 0 & 1 & -1 & 0 & -100 \\ 1 & -1 & 0 & 0 & 0 & 400 \\ 0 & 1 & -1 & 0 & 0 & -400 \end{bmatrix}$$

ref $\rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & -400 \\ 0 & 1 & 0 & -1 & 0 & -500 \\ 0 & 0 & 1 & -1 & 0 & -100 \\ 0 & 0 & 0 & 0 & 1 & 500 \end{bmatrix}$

$A = D - 100$
 $B = D - 500$
 $C = D - 100$
 $D = D$
 $E = 500$

b. $X_1 + X_2 = 20$
 $X_3 + 20 = X_4$
 $X_1 + 10 = X_5$
 $X_5 + 10 = X_4$
 $X_2 + X_3 = 20$

$X_1 + X_2 = 20$
 $X_3 - X_4 = -20$
 $X_1 - X_5 = -10$
 $-X_4 + X_5 = -10$
 $X_2 + X_3 = 20$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & 1 & -1 & 0 & -20 \\ 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 0 & 0 & -1 & 1 & -10 \\ 0 & 1 & 1 & 0 & 0 & 20 \end{bmatrix}$$

ref $\rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$X_1 - X_5 = -10$
 $X_2 + X_5 = 30$
 $X_3 - X_5 = -10$
 $X_4 - X_5 = 10$

$X_1 = X_5 - 10$
 $X_2 = -X_5 + 30$
 $X_3 = X_5 - 10$
 $X_4 = X_5 + 10$
 $X_5 = X_5$

c. $400 + X_2 = X_1$
 $X_1 + X_3 = 600 + X_4$
 $300 = X_2 + X_3 + X_5$
 $X_4 + X_5 = 100$

$-X_1 + X_2 = -400$
 $X_1 + X_3 - X_4 = 600$
 $X_2 + X_3 + X_5 = 300$
 $X_4 + X_5 = 100$

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 & -400 \\ 1 & 0 & 1 & -1 & 0 & 600 \\ 0 & 1 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & 1 & 100 \end{bmatrix}$$

ref $\rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 700 \\ 0 & 1 & 1 & 0 & 1 & 300 \\ 0 & 0 & 0 & 1 & 1 & 100 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$X_1 + X_3 + X_5 = 700$
 $X_2 + X_3 + X_5 = 300$
 $X_4 + X_5 = 100$

$X_1 = -X_3 - X_5 + 700$
 $X_2 = -X_3 - X_5 + 300$
 $X_3 = X_3$
 $X_4 = -X_5 + 100$
 $X_5 = X_5$

5a. i. $A + B = \begin{bmatrix} 12 & 4 \\ 0 & 4 \end{bmatrix}$

ii. $2B - 3C = \begin{bmatrix} 18 & 6 \\ 2 & 0 \end{bmatrix} - \begin{bmatrix} 6 & -6 \\ -12 & 12 \end{bmatrix} = \begin{bmatrix} 12 & 12 \\ 14 & -12 \end{bmatrix}$

b. $AA = \begin{bmatrix} 12 & 4 \\ -4 & 16 \end{bmatrix}$

iii. $-5A = \begin{bmatrix} -5 & 0 & 10 \end{bmatrix}$

c. i. $AB = \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 9 & 3 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 27+1 & 9+0 \\ -9+4 & -3+0 \end{bmatrix} = \begin{bmatrix} 28 & 9 \\ -5 & -3 \end{bmatrix}$

Sc. cont'd.

(4)

$$ii. DE = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 0 & -3 & 5 \\ 1 & -4 & 0 \\ -1 & 2 & -7 \end{bmatrix} = \begin{bmatrix} -1 & -7 & -23 \\ 1 & 2 & -10 \\ -5 & 9 & 8 \end{bmatrix}$$

$$iii. GC = \begin{bmatrix} 0 & -7 \\ 11 & -5 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -4 & 4 \end{bmatrix} = \begin{bmatrix} 40 & -40 \\ 42 & -42 \\ -8 & 8 \end{bmatrix}$$

iv. EH = not defined $(3 \times 3) \times (1 \times 3)$

v. BA = $\begin{bmatrix} 9 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ -1 & 4 \end{bmatrix}$ mismatch $\begin{bmatrix} 24 & 21 \\ 3 & 1 \end{bmatrix}$

$$vi. BF = \begin{bmatrix} 9 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 3 & -2 & 0 \\ 2 & 4 & -1 & 5 \end{bmatrix} = \begin{bmatrix} 15 & 39 & -21 & 15 \\ 1 & 3 & -2 & 0 \end{bmatrix}$$

$$vii. DG = \begin{bmatrix} 1 & 3 & 4 \\ -2 & 1 & 0 \\ 3 & -4 & 1 \end{bmatrix} \begin{bmatrix} 6 & -7 \\ 11 & -5 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 47 & -10 \\ -1 & 9 \\ -24 & 2 \end{bmatrix}$$

$$viii. BJ = \begin{bmatrix} 9 & 3 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 33 \\ 4 \end{bmatrix}$$

d. i. $A^{-1} = \begin{bmatrix} 4/13 & -1/13 \\ 1/13 & 3/13 \end{bmatrix}$

iii. $C^{-1} =$ no inverse
Singular

ii. $B^{-1} = \begin{bmatrix} 0 & 1 \\ 1/3 & -3 \end{bmatrix}$

iii. $D^{-1} = \begin{bmatrix} 1/27 & -19/27 & -4/27 \\ 2/27 & -11/27 & -8/27 \\ 5/27 & 13/27 & 7/27 \end{bmatrix}$

iv. $E^{-1} = \begin{bmatrix} -28/31 & 1/31 & -20/31 \\ -7/31 & -9/31 & -5/31 \\ 2/31 & -3/31 & -3/31 \end{bmatrix}$

e. i. $\det A = 13$ $(12+1)$

ii. $\det B = -3$ $(0-3)$

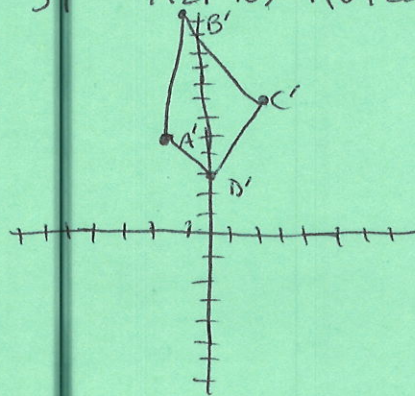
iii. $\det C = 0$ $(8-8)$

iv. $\det D = 27$ $4(8-3)+1(1+6)$

v. $\det E = -31$ $-1(21-10)-1(0+20)$

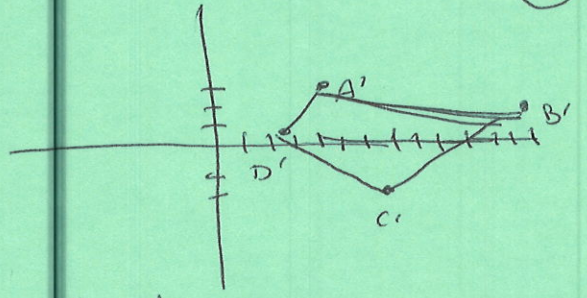
6. a. pts. $A(2,5), B(0,9), C(7,2), D(2,2)$

$$A \begin{bmatrix} 2 \\ 5 \end{bmatrix} = A \begin{bmatrix} 10 \\ 9 \end{bmatrix} = A \begin{bmatrix} 7 \\ 2 \end{bmatrix} = A \begin{bmatrix} 2 \\ 2 \end{bmatrix} =$$

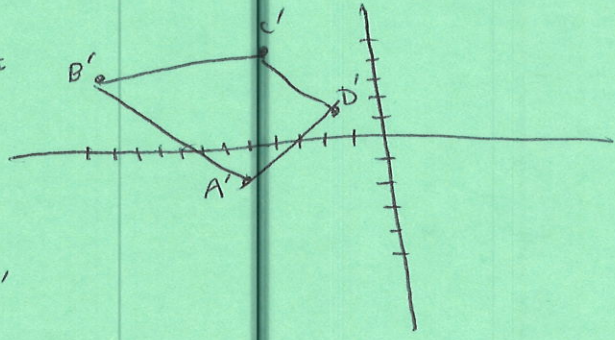


6a cont'd

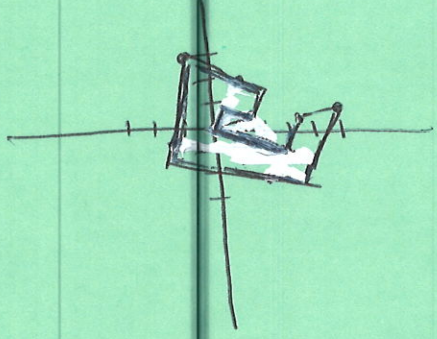
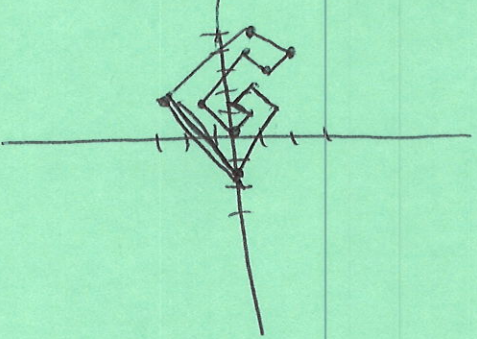
$$B \begin{bmatrix} 2 \\ 5 \end{bmatrix} = B \begin{bmatrix} 10 \\ 9 \end{bmatrix} = B \begin{bmatrix} 7 \\ 2 \end{bmatrix} = B \begin{bmatrix} 2 \\ 2 \end{bmatrix} =$$



$$C \begin{bmatrix} 2 \\ 5 \end{bmatrix} = C \begin{bmatrix} 10 \\ 9 \end{bmatrix} = C \begin{bmatrix} 7 \\ 2 \end{bmatrix} = C \begin{bmatrix} 2 \\ 2 \end{bmatrix} =$$



6b. Points D(-1,-1), E(-1,3), F(4,3), G(4,1), H(3,1), I(3,2), J(0,2), K(0,0), L(1,0), M(1,1), N(2,1), O(2,-1)



$$7a = 1a$$

$$A = \begin{bmatrix} 3 & 6 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} -3 \\ 10 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -3 & 6 \\ 10 & 7 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 3 & -3 \\ 5 & 10 \end{bmatrix}$$

$$\det A = -9$$

$$\det A_1 = -81$$

$$\det A_2 = 45$$

$$x_1 = \frac{-81}{-9} = 9$$

$$x_2 = \frac{45}{-9} = -5$$

7b = 1b

$$A = \begin{bmatrix} 2 & 0 & 0 & -4 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 1 & 4 \\ -3 & 2 & 3 & 1 \end{bmatrix}$$

det A = -54

$$A_1 = \begin{bmatrix} -10 & 0 & 0 & -4 \\ 0 & 3 & 3 & 0 \\ -1 & 0 & 1 & 4 \\ 5 & 2 & 3 & 1 \end{bmatrix}$$

det A₁ = 162

$$A_2 = \begin{bmatrix} 2 & -6 & 0 & -4 \\ 0 & 0 & 3 & 0 \\ 0 & 1 & 3 & 4 \\ -3 & 5 & 3 & 1 \end{bmatrix}$$

det A₂ = -270

$$A_3 = \begin{bmatrix} 2 & 0 & -10 & -4 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & -1 & 4 \\ -3 & 2 & 5 & 1 \end{bmatrix}$$

det A₃ = 270

$$A_4 = \begin{bmatrix} 2 & -10 & 0 & -10 \\ 0 & 0 & 3 & 0 \\ 0 & -1 & 1 & -1 \\ -3 & 5 & 3 & 5 \end{bmatrix}$$

det A₄ = -54

x₁ = $\frac{162}{-54} = -3$ x₂ = 5 x₃ = -5 x₄ = -1

7c = 1c

$$A = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & 2 \\ 2 & -4 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

det A = 15

$$A_1 = \begin{bmatrix} 0 & -2 & -3 \\ 0 & 1 & 2 \\ 0 & -4 & 9 \end{bmatrix}$$

det A₁ = 0

$$A_2 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 2 \\ 2 & 0 & 9 \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 2 & -4 & 0 \end{bmatrix}$$

det A₂ = 0 det A₃ = 0

x₁ = x₂ = x₃ = 0

7d = 2a

$$A = \begin{bmatrix} 12 & -12 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 7 \\ 0 \end{bmatrix}$$

det A = 84

$$A_1 = \begin{bmatrix} 7 & -12 \\ 0 & 4 \end{bmatrix}$$

det A₁ = 28

$$A_2 = \begin{bmatrix} 12 & 7 \\ 3 & 0 \end{bmatrix}$$

det A₂ = -21

x₁ = $\frac{28}{84} = \frac{1}{3} \Rightarrow x = 3$ x₂ = $-\frac{1}{4} \Rightarrow y = 4$

7e = 2b

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 3 & -4 & 0 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix}$$

det A = -55

$$A_1 = \begin{bmatrix} 5 & 1 & -2 \\ -1 & -4 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

det A₁ = -55

$$A_2 = \begin{bmatrix} 2 & 5 & -2 \\ 3 & -1 & 0 \\ 2 & 0 & 3 \end{bmatrix} \quad A_3 = \begin{bmatrix} 2 & 1 & 5 \\ 3 & -4 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

det A₂ = -55 det A₃ = 55

x₁ = 1, x₂ = 1, x₃ = -1

8a=1a

$$A = \begin{bmatrix} 3 & 6 \\ 5 & 7 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} -7/9 & 2/3 \\ 5/9 & -1/3 \end{bmatrix} \quad A^{-1} \begin{bmatrix} -3 \\ 10 \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$$

8b=1b

$$A = \begin{bmatrix} 2 & 0 & 0 & -4 \\ 0 & 3 & 3 & 0 \\ 0 & 0 & 1 & 4 \\ -3 & 2 & 3 & 1 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 1/6 & 4/27 & 2/9 & -2/9 \\ -2/3 & 17/27 & -5/9 & 4/9 \\ 2/3 & -8/27 & 5/9 & 4/9 \\ -1/6 & 2/27 & 1/9 & -1/9 \end{bmatrix} \quad A^{-1} \begin{bmatrix} -10 \\ 0 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ -5 \\ 1 \end{bmatrix}$$

8c=1c

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 1 & 2 \\ 2 & -4 & 9 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 17/3 & 2 & -7/3 \\ 2/3 & 1 & -2/3 \\ -2/3 & 0 & 1/3 \end{bmatrix} \quad A^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

8d=2a

$$A = \begin{bmatrix} 12 & -12 \\ 3 & 4 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 1/21 & 1/7 \\ -1/28 & 1/7 \end{bmatrix} \quad A^{-1} \begin{bmatrix} 7 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/4 \end{bmatrix} \rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

8e=2b

$$A = \begin{bmatrix} 2 & 1 & -2 \\ 3 & -4 & 0 \\ 2 & 1 & 3 \end{bmatrix} \quad A^{-1} = \begin{bmatrix} 12/55 & 1/11 & 8/55 \\ 9/55 & -2/11 & 6/55 \\ -1/55 & 0 & 1/5 \end{bmatrix} \quad A^{-1} \begin{bmatrix} 5 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$