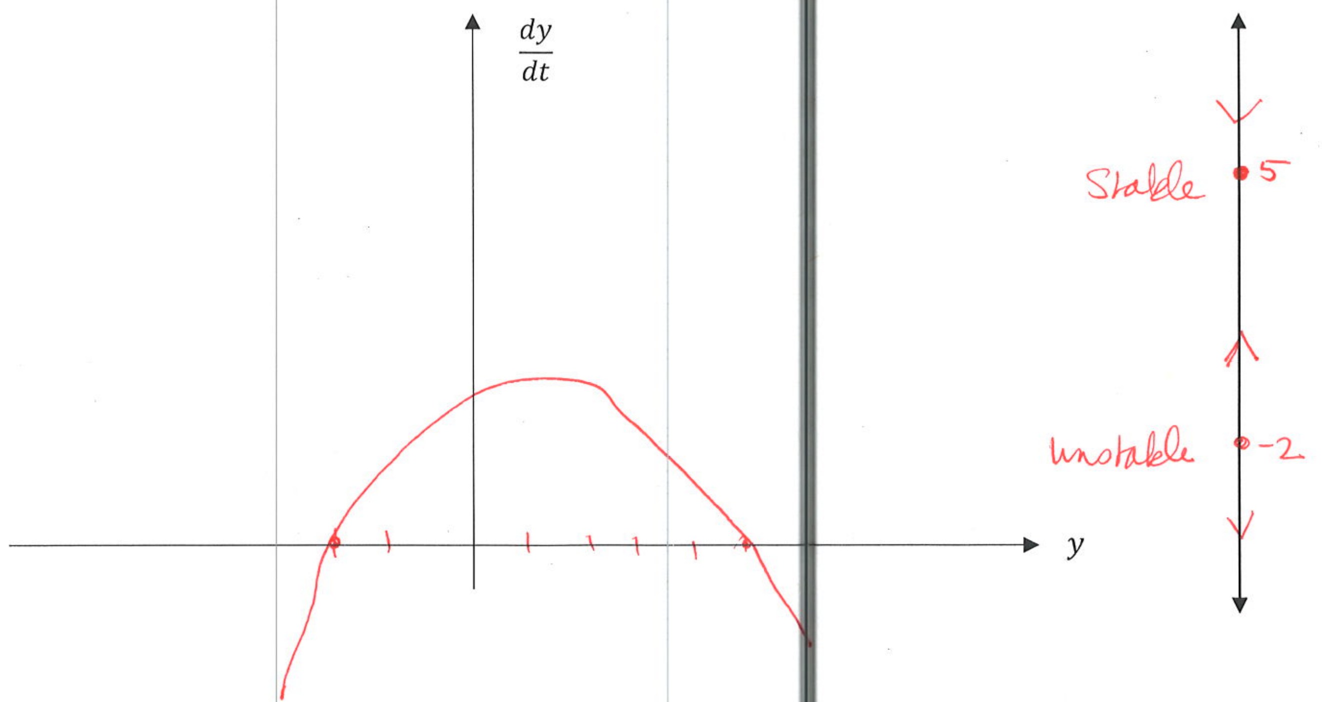
MTH 291, Final Exam, Fall 2017 Name KEY

Instructions: Show all work. You will earn full credit for correct answers only when accompanied by work or explanation. Answers that are incorrect and have no work will not receive any partial credit. Use exact answers, except in applied problems: round to two decimal places, or the number requested in the problem.

1. Sketch the phase plane of $\frac{dy}{dt} = 10 + 3y - y^2$ on the axis below, and then convert that to a phase line. Describe the stability of each point. (15 points) $-(y^2 - 3y - 10) = -(y-5)(y+2)$



2. Solve the differential equation $\frac{dy}{dx} = x\sqrt{1 - y^2}$. (15 points)

$$\frac{dy}{\sqrt{1-y^2}} = x\,dx$$

$$\arcsin y = \frac{1}{2}x^2 + C$$

$$y = \sin\left(\frac{x^2}{2} + C\right)$$

3. Solve the differential equation $\frac{dy}{dx} = 2y + x^2 + 5$ by the method of integrating factors (reverse product rule). (15 points)

$$\mu = e^{\int -2\,dx} = e^{-2x}$$

$$\frac{dy}{dx} - 2y = x^2+5$$

$$e^{-2x}y' - 2e^{-2x}y = (x^2+5)e^{-2x}$$

$$\int (e^{-2x}y)' = \int (x^2+5)e^{-2x} + C$$

$$e^{-2x}y = -\frac{1}{4}e^{-2x}(2x^2+2x+11) + C$$

$$y = -\frac{1}{4}(2x^2+2x+11) + Ce^{2x}$$

4. Solve the second order differential equation $3y'' + 2y' + y = 0$ for the general solution. (15 points)

$$3r^2+2r+1=0$$

$$r = \frac{-2 \pm \sqrt{4-12}}{2(3)} = \frac{-2 \pm \sqrt{8}\,i}{6} = -\frac{1}{3} \pm \frac{\sqrt{2}}{3}i$$

$2\sqrt{2}$

$$y_1 = c_1 e^{-\frac{1}{3}t}\cos\left(\frac{\sqrt{2}}{3}t\right) + c_2 e^{-\frac{1}{3}t}\sin\left(\frac{\sqrt{2}}{3}t\right)$$

5. Solve the second order differential equation $y'' - 10y' + 25y = 0$ for the general solution. (15 points)

$$r^2 - 10r + 25 = 0$$

$$(r-5)^2 = 0$$

$$r = 5$$

$$y = c_1 e^{5t} + c_2 t e^{5t}$$

6. Identify the Ansatz to find the particular solution for the differential equation $y'' + 4y = f(x)$. (5 points each)

a. $f(x) = x^2 - 2x$

$y_1 = \cos 2x$
$y_2 = \sin 2x$

$Y(x) = Ax^2 + Bx + C$

b. $f(x) = 2e^{4x}$

$Y(x) = Ae^{4x}$

c. $f(x) = 3\sin 2x$

$Y(x) = Ax\cos 2x + Bx\sin 2x$

7. Find the general solutions to the systems. (15 points each)

a. $\vec{x}' = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \vec{x}$

$(2-\lambda)(-1-\lambda) - 1 = 0$

$\lambda^2 - \lambda - 2 - 1 = 0$

$\lambda^2 - \lambda - 3 = 0$

$\lambda = \dfrac{1 \pm \sqrt{1+12}}{2}$

$= \dfrac{1}{2} \pm \dfrac{\sqrt{13}}{2}$

$\begin{bmatrix} 2-(\frac{1}{2}+\frac{\sqrt{13}}{2}) & 1 \\ 1 & -1-(\frac{1}{2}+\frac{\sqrt{13}}{2}) \end{bmatrix} =$

$\begin{bmatrix} 3/2 - \frac{\sqrt{13}}{2} & 1 \\ 1 & -3/2 - \frac{\sqrt{13}}{2} \end{bmatrix}$

$x_1 - (3/2 + \frac{\sqrt{13}}{2}) x_2 = 0$

$x_1 = \left(\frac{3}{2} + \frac{\sqrt{13}}{2}\right) x_2$

$\vec{v}_1 = \begin{bmatrix} 3+\sqrt{13} \\ 2 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 3-\sqrt{13} \\ 2 \end{bmatrix}$

$\vec{X} = c_1 \begin{bmatrix} 3+\sqrt{13} \\ 2 \end{bmatrix} e^{\left(\frac{1+\sqrt{13}}{2}\right)t} + c_2 \begin{bmatrix} 3-\sqrt{13} \\ 2 \end{bmatrix} e^{\left(\frac{1-\sqrt{13}}{2}\right)t}$

b. $\vec{x}' = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix} \vec{x}$

$(1-\lambda)(-3-\lambda)+8=0$

$\lambda^2+2\lambda-3+8=0$

$\lambda^2+2\lambda+5=0$

$\lambda = \frac{-2 \pm \sqrt{4-20}}{2}$

$= -1 \pm 2i$

$\begin{bmatrix} 1-(-1+2i) & -8 \\ 1 & -3-(-1+2i) \end{bmatrix}$

$\begin{bmatrix} 2-2i & -8 \\ 1 & -2-2i \end{bmatrix}$

$x_1 - (2+2i)x_2 = 0$

$x_1 = (2+2i)x_2$

$\vec{v}_1 = \begin{bmatrix} 2+2i \\ 1 \end{bmatrix}$

$e^{-t}\begin{bmatrix} 2+2i \\ 1 \end{bmatrix}(\cos 2t + i\sin 2t) =$

$e^{-t}\begin{bmatrix} 2\cos 2t + 2i\sin 2t + 2i\cos 2t - 2\sin 2t \\ \cos 2t + i\sin 2t \end{bmatrix}$

$\vec{x} = c_1\begin{bmatrix} 2\cos 2t - 2\sin 2t \\ \cos 2t \end{bmatrix} e^{-t} + c_2\begin{bmatrix} 2\sin 2t + 2\cos 2t \\ \sin 2t \end{bmatrix} e^{-t}$

8. Use Euler's method for two steps by hand to estimate the value of $y(1.5)$ if $y' = 2x - 3y + 1$ and $y(1) = 5$. (15 points)

$\Delta t = .5/2 = .25$

$n=0$ $x_0 = 1$ $y_0 = 5$ $m_0 = 2(1) - 3(5) + 1 = -12$ $\Delta t = .25$ $y_1 = .25(-12)+5$

$= -3+5 = 2$

$n=1$ $x_1 = 1.25$ $y_1 = 2$ $m_1 = 2(1.25) - 3(2) + 1 = -2.5$

$y_2 = .25(-2.5) + 2$

$= 1.375$

$n=2$ $x_2 = 1.5$ $y_2 = 1.375$

$y(1.5) \approx 1.375$

MTH 291, Final Exam Supplement, Fall 2017 Name KEY

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1. Consider the differential equation $y' = y^2 + 4$. (4 points each)
   a. Explain why there are no constant solutions of the differential equation.

   there is no real solution to $y' = 0$ / $0 = y^2 + 4$

   b. Describe the graph of the solution $y(x)$ (for example, can a solution to the curve have any relative extrema?).

   there can be no extrema, since y' can never be zero, but there will be an inflection point at $y = 0$

   c. Explain why $y = 0$ is the y-coordinate of a point of inflection on the solution curve.

   $y' = y^2 + 4$

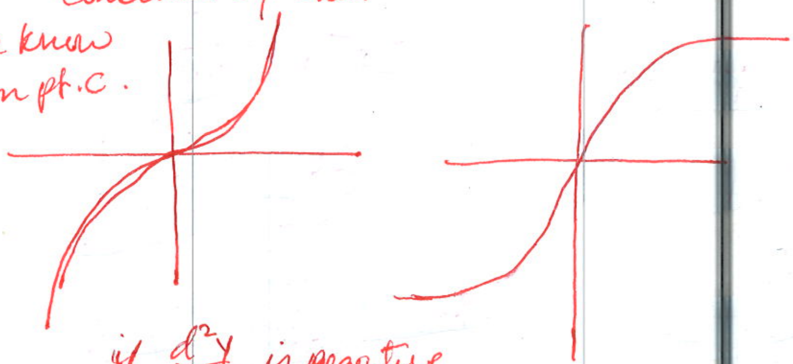
   implicit differentiation

   $y'' = \frac{d^2y}{dt^2} = 2y\frac{dy}{dt} = 2y(y^2+4) = 0$

   $y = 0$ is a solution

d. Sketch the graph of a solution $y(x)$ of the differential equation who shape is suggested by the above.

consistent w/ what we know from pt. c.



if $\frac{d^2y}{dt^2}$ is negative for $y<0$

if $\frac{d^2y}{dt^2}$ is negative for $y>0$ 2nd graph is appropriate not consistent w/ equation

e. Solve the differential equation (by separation). Does it confirm the results above? Explain.

$$\frac{dy}{dt} = y^2+4$$

$$\frac{dy}{y^2+4} = dt$$

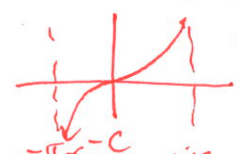
$$\frac{1}{2}\arctan(2y) = t+C$$

$$\arctan(2y) = 2t+C$$

$$2y = \tan(2t+C)$$

$$y = \frac{1}{2}\tan(2t+C)$$

yes since this graph is



always increasing inflection at 0

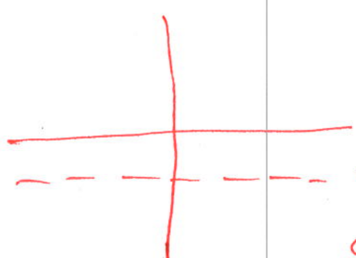
2. Determine a region in the xy-plane where $(1+y^3)y' = x^2$ is defined, and guaranteed to have a unique solution. Sketch the region. (10 points)

$$y' = \frac{x^2}{1+y^3} = f$$

$y^3+1=0$ when $y=-1$. not defined

$$\frac{\partial f}{\partial y} = -1(1+y^3)^{-2}3y^2$$

$$= \frac{-3y^2}{(1+y^3)^2}$$



$y=-1$

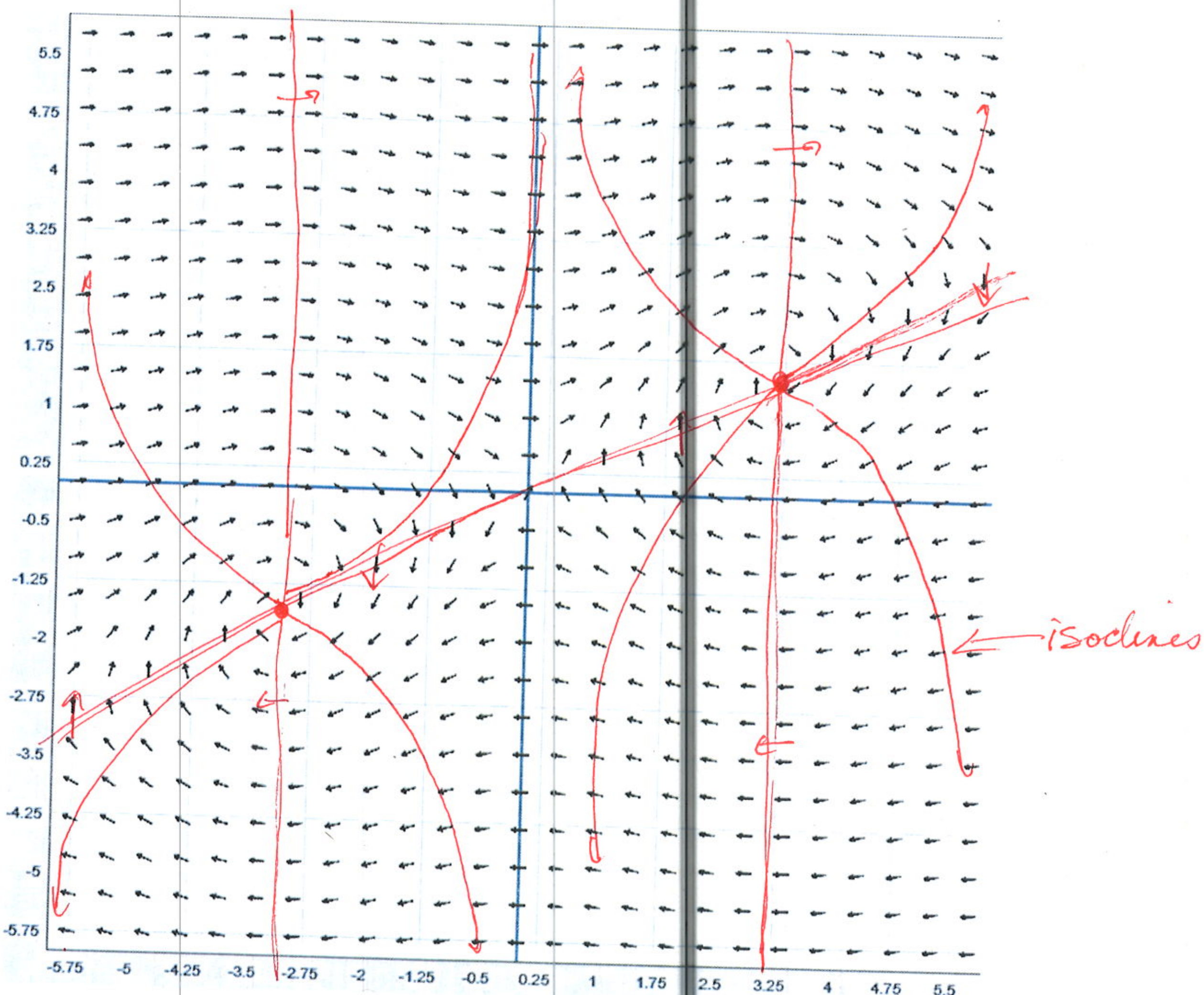
defined everywhere else

3. The phase plane of $\begin{cases} \dfrac{dx}{dt} = y - \dfrac{1}{2}x \\ \dfrac{dy}{dt} = \sin x \end{cases}$ is shown to the below. (15 points)

a. Sketch on the graph (and label) the approximate location of the nullclines.

b. Isoclines are curves where $f(x, y) = c$, for some constant c. They have the property of being perpendicular to the vector field. Sketch and label at least three isoclines.

c. Identify any equilibria.



equilibria $(-\pi, -\pi/2)$ and $(\pi, \pi/2)$

4. Discuss how the method of undetermined coefficients can be used to solve the differential equation $y'' + y = \sin x \cos 2x$. (10 points)

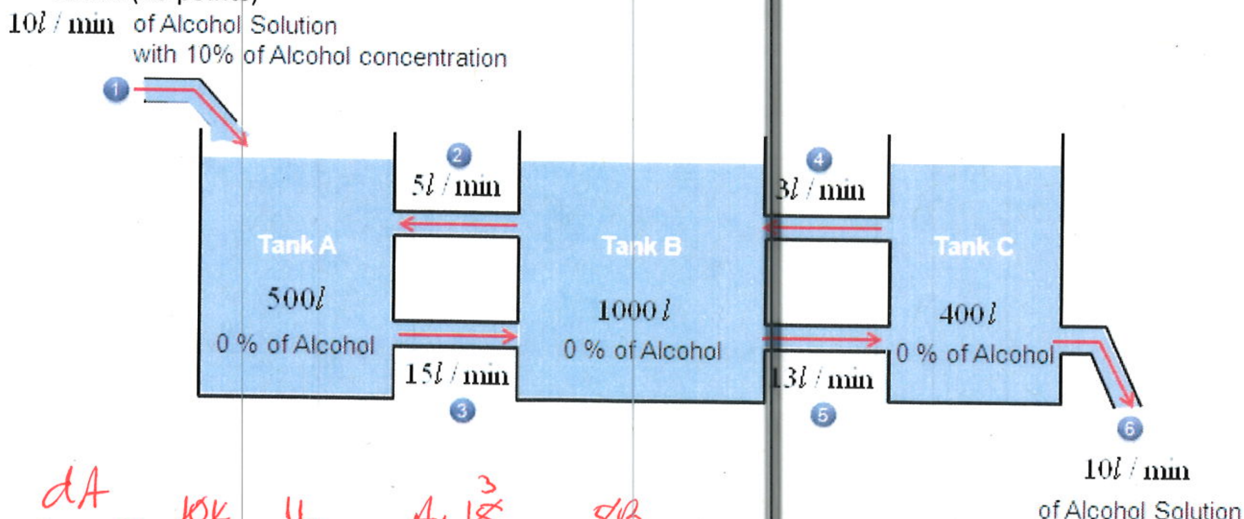
apply the identity $\sin(a)\cos(b) = \frac{1}{2}(\sin(a+b) + \sin(a-b))$

to get $y'' + y = \frac{1}{2}(\sin(3x) + \sin(-x)) \Rightarrow$

$y'' + y = \frac{1}{2}\sin 3x - \frac{1}{2}\sin x$

and apply undetermined coefficients to the result.

5. Set up a system of linear differential equations to model the coupled tank problem below. Do not solve. (10 points)



$$\frac{dA}{dt} = \frac{10\cancel{L}}{min}\cdot\frac{1L}{\cancel{10L}} - \frac{A\cdot\cancel{15}^{3}}{\cancel{500}_{100}} + \frac{\cancel{8B}}{\cancel{1000}_{200}}$$

$$\frac{dB}{dt} = \frac{\cancel{15}^{3}A}{\cancel{500}_{100}} - \frac{\cancel{8}B}{1000} - \frac{13B}{1000} + \frac{3C}{400}$$

$\frac{-\cancel{18}B}{1000}$

$$\frac{dC}{dt} = \frac{13B}{1000} - \frac{3C}{400} - \frac{C\cdot 10}{400}$$

$\Rightarrow$

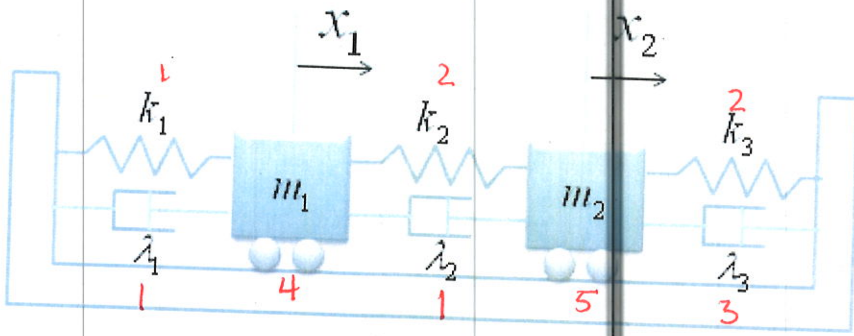
$$\frac{dA}{dt} = -\frac{3A}{100} + \frac{B}{200} + 1$$

$$\frac{dB}{dt} = \frac{3A}{100} - \frac{9B}{500} + \frac{3C}{400}$$

$$\frac{dC}{dt} \quad \frac{13B}{1000} - \frac{13C}{400}$$

$A(0) = B(0) = C(0) = 0$

6. Set up a system of differential equations (first or second order) that models the coupled spring-mass system below, where $k_1 = 1, k_2 = k_3 = 2, \lambda_1 = \lambda_2 = 1, \lambda_3 = 3, m_1 = 4, m_2 = 5$. (10 points)



$$\underset{4}{m_1}\frac{d^2x_1}{dt^2} = -\underbrace{(\lambda_1+\lambda_2)}_{2}x_1' - \underbrace{(k_1+k_2)}_{3}x_1 + \underbrace{\lambda_1}_{1}x_2' + \underbrace{k_2}_{2}x_2$$

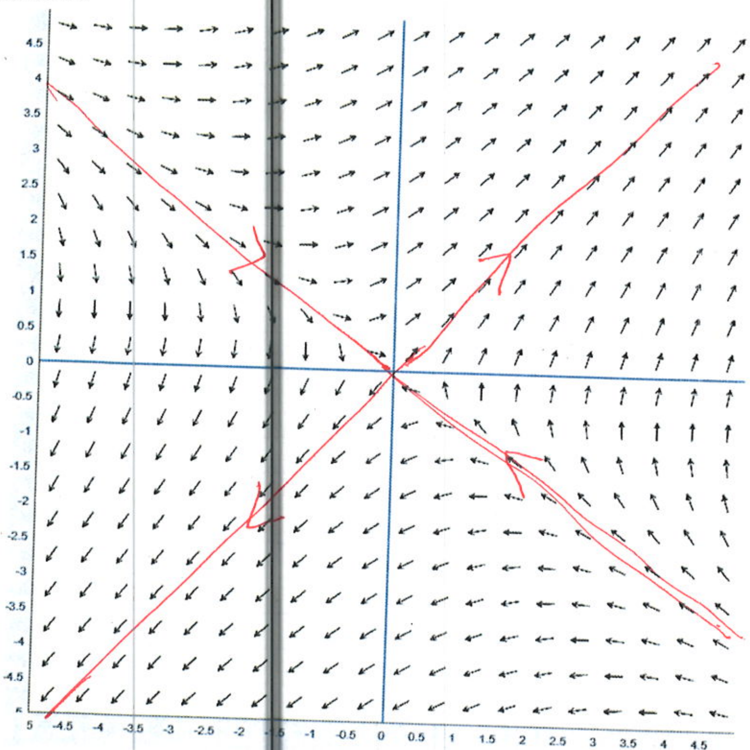
$$\underset{5}{m_2}\frac{d^2x_2}{dt^2} = -\underbrace{(\lambda_2+\lambda_3)}_{4}x_2' - \underbrace{(k_2+k_3)}_{4}x_2 + \underbrace{\lambda_2}_{1}x_1' + \underbrace{k_2}_{2}x_1$$

$$\begin{cases} \frac{d^2x_1}{dt^2} = -\frac{2}{4}x_1' - \frac{3}{4}x_1 + x_2' + 2x_2 \\ \frac{d^2x_2}{dt^2} = -\frac{4}{5}x_2' - \frac{4}{5}x_2 + x_1' + 2x_1 \end{cases}$$

7. Consider the system of differential equations:

$$\begin{cases} \frac{dx}{dt} = x + 5y \\ \frac{dy}{dt} = 4x + 3y \end{cases}$$

The phase plane is shown to the right. Identify the straight-line solutions and characterize the equilibrium as an attractor, repeller, or saddle point. (10 points)



Saddle point

8. Verify that the solution of $\vec{x}' = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}\vec{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} t^2 + \begin{bmatrix} 4 \\ -6 \end{bmatrix} t + \begin{bmatrix} -1 \\ 5 \end{bmatrix}$ is $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ -1-\sqrt{2} \end{bmatrix} e^{\sqrt{2}t} + c_2 \begin{bmatrix} 1 \\ -1+\sqrt{2} \end{bmatrix} e^{-\sqrt{2}t} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t^2 + \begin{bmatrix} -2 \\ 4 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. (10 points)

$$X'(t) = c_1 \begin{bmatrix} \sqrt{2} \\ -\sqrt{2}-2 \end{bmatrix} e^{\sqrt{2}t} + c_2 \begin{bmatrix} -\sqrt{2} \\ \sqrt{2}-2 \end{bmatrix} e^{-\sqrt{2}t} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} t + \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} c_1\sqrt{2}e^{\sqrt{2}t} + c_2(-\sqrt{2})e^{-\sqrt{2}t} + 2t - 2 \\ c_1(-\sqrt{2}-2)e^{\sqrt{2}t} + c_2(\sqrt{2}-2)e^{-\sqrt{2}t} + 4 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t} + t^2 - 2t + 1 \\ c_1(-1-\sqrt{2})e^{\sqrt{2}t} + c_2(-1+\sqrt{2})e^{-\sqrt{2}t} + 4t \end{bmatrix} + \begin{bmatrix} t^2 \\ t^2 \end{bmatrix} + \begin{bmatrix} 4t \\ -6t \end{bmatrix} + \begin{bmatrix} -1 \\ 5 \end{bmatrix} =$$

$$= \begin{bmatrix} -\cancel{c_1 e^{\sqrt{2}t}} - \cancel{c_2} e^{-\sqrt{2}t} - \cancel{t^2} + 2t - 1 + \cancel{t^2} + \cancel{4t} - 1 + c_1(\cancel{1}+\sqrt{2})e^{\sqrt{2}t} + c_2(\cancel{1}-\sqrt{2})e^{-\sqrt{2}t} - \cancel{4t} \\ -c_1 e^{\sqrt{2}t} - c_2 e^{-\sqrt{2}t} - \cancel{t^2} + \cancel{2t} - 1 + \cancel{t^2} - \cancel{6t} + 5 + c_1(-1-\sqrt{2})e^{\sqrt{2}t} + c_2(-1+\sqrt{2})e^{-\sqrt{2}t} + \cancel{4t} \end{bmatrix}$$

$$= \begin{bmatrix} c_1\sqrt{2}e^{\sqrt{2}t} - \sqrt{2}e^{-\sqrt{2}t} + 2t - 2 \\ c_1(-2-\sqrt{2})e^{\sqrt{2}t} + c_2(\sqrt{2}-2)e^{-\sqrt{2}t} + 4 \end{bmatrix}$$ agrees, so it is a solution

9. Give examples second-order equations for a spring system with the following properties: (4 points each)

a. Undamped

$y'' + 4y = 0$

b. Underdamped

$y'' + 2y' + 4y = 0$

c. Overdamped

$y'' + 5y' + 4y = 0$

d. Exhibits resonance

$y'' + 4y = \sin 2t$

e. Exhibits beat phenomena

$y'' + 64y = \sin 7t$

answers will vary