MTH 291 Skills Problems #3 Key

1. it is linear since $\frac{d^3y}{dt^3}, \frac{dy}{dt}, y$ are not raised to powers, multiply each other, etc. 3rd order. Ordinary.

2. a. $y' - 2y = 3e^t$ $\quad \mu = e^{\int 2\,dt} = e^{-2t}$

$e^{-2t}y' - 2e^{-2t}y = 3e^te^{-2t} = 3e^{-t}$

$\int (ye^{-2t})' = \int 3e^{-t}\,dt \rightarrow ye^{-2t} = -3e^{-t} + C$

$y = -3e^{-t} + Ce^{-2t}$

b. $ty' - y = -t^2e^{-t} \rightarrow y' - \frac{1}{t}y = te^{-t}$

$\frac{1}{t}y' - \frac{1}{t^2}y = e^{-t}$ $\quad \mu = e^{\int -\frac{1}{t}dt} = e^{-\ln t} = \cancel{e}^{\cancel{\ln} \frac{1}{t}} = \frac{1}{t}$

$\int \left(y\cdot\frac{1}{t}\right)' = \int e^{-t}\,dt \rightarrow y\cdot\frac{1}{t} = -e^{-t} + C$

$y = -te^{-t} + Ct$

c. $t^3y' + 4t^2y = e^{-t}$ $\quad y(-1) = 0$

$y' + \frac{4}{t}y = \frac{e^{-t}}{t^3}$ $\quad \mu = e^{\int \frac{4}{t}dt} = e^{4\ln t} = \cancel{e}^{\cancel{\ln} t^4} = t^4$

$t^4y' + 4t^3y = te^{-t}$

$\int (t^4y)' = \int te^{-t}\,dt$

$u = t$
$du = dt$
$dv = e^{-t}$
$v = -e^{-t}$

$t^4y = -te^{-t} + \int e^{-t}\,dt$
$= -te^{-t} - e^{-t} + C$

$y = \frac{-e^{-t}}{t^3} - \frac{e^{-t}}{t^4} + \frac{C}{t^4}$

$0 = \frac{+e}{-1} - \frac{e}{1} + \frac{C}{1} \Rightarrow C = 0$

$y = \frac{-e^{-t}}{t^3} - \frac{e^{-t}}{t^4}$

②

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d. $ty' + 2y = \sin t \rightarrow y' + \frac{2}{t} y = \frac{\sin t}{t}$

$\mu = e^{\int \frac{2}{t} dt} = e^{2\ln t} = \cancel{e}^{\cancel{\ln} t^2} = t^2$

$t^2 y' + 2t y = t \sin t \rightarrow \int (yt^2)' = \int t \sin t \, dt$

$u = t \quad dv = \sin t$

$du = dt \quad v = -\cos t$

$t^2 y = -t\cos t + \int \cos t \, dt = -t\cos t + \sin t + C$

$y = -\frac{\cos t}{t} + \frac{\sin t}{t^2} + \frac{C}{t^2}$

e. $y' - 2y = e^{2t} \quad y(0) = 2 \quad \mu = e^{\int -2dt} = e^{-2t}$

$e^{-2t} y' - 2e^{-2t} y = 1 \rightarrow \int \left(e^{-2t} y\right)' = \int 1 dt \Rightarrow$

$e^{-2t} y = t + C \rightarrow y = te^{2t} + Ce^{2t}$

$2 = \cancel{0e^{0}} + Ce^{0 \nearrow 1} \quad C = 2 \quad y = te^{2t} + 2e^{2t}$

3. This formula is derived from method of integrating factors so most of the steps are the same and the solutions are the same.

4. $A(0) = 30 \qquad \frac{dA}{dt} = \frac{4\,l}{m} \cdot \frac{1\,gr}{l} - \frac{5\,l}{m} \cdot \frac{A}{200 - t}$

$\frac{dA}{dt} = 4 - \frac{5A}{200 - t}$

4 cont'd

$$\frac{dA}{dt} + \frac{5A}{200-t} = 4 \qquad \mu = e^{\int 5/200-t\,dt} = e^{-5\ln(200-t)} = \cancel{e}^{\cancel{\ln}(200-t)^{-5}} = (200-t)^{-5}$$

$$(200-t)^{-5}\frac{dA}{dt} + 5(200-t)^{-6}A = 4(200-t)^{-5}$$

$$\int\left[(200-t)^{-5}A\right]' = \int 4(200-t)^{-5}\,dt$$

$$(200-t)^{-5}A = \frac{4(200-t)^{-4}}{+4} + C$$

$$A = +(200-t) + C(200-t)^{5}$$

$$30 = (200) + C(200)^{5}$$

$$C = -5.3125\times10^{-10}$$

$$A = 200-t - 5.3125\times10^{-10}(200-t)^{5}$$

5. $t^2y' + 2ty = y^3 \rightarrow y' + \frac{2}{t}y = \frac{y^3}{t^2}$ $\qquad n=3$

$(1-n)y^{-n} = -2y^{-3}$

$$-2y^{-3}y' - \frac{4}{t}y^{-2} = -\frac{2}{t^{2}}$$

$z = y^{-2}$

$$\frac{dz}{dt} - \frac{4}{t}z = -\frac{2}{t^2} \qquad \frac{dz}{dt} = -2y^{-3}y'$$

$$\mu = e^{\int -\frac{4}{t}dt} = e^{-4\ln t} = \cancel{e}^{\cancel{\ln} t^{-4}} = t^{-4}$$

$$t^{-4}\frac{dz}{dt} - 4t^{-5}z = -2t^{-2}$$

$$\int(t^{-4}z)' = \int -2t^{-2}\,dt = \frac{2}{t} + C \rightarrow z = 2t^3 + Ct^4$$

5 cont'd

$y^{-2} = 2t^3 + Ct^4 \Rightarrow \frac{1}{y^2} = 2t^3 + Ct^4$

$y^2 = \frac{1}{2t^3 + Ct^4}$ $\quad y = \pm\sqrt{\frac{1}{2t^3+Ct^4}}$

6. $t^2 y'' + 2ty' - 2y = 0$ $\quad y_1 = t$

a.

$y_2 = t \cdot v$

$y_2' = v + tv'$

$y_2'' = 2v' + tv''$

$t^2(2v' + tv'') + 2t(v + tv') - 2tv = 0$

$2t^2v' + t^3v'' + \cancel{2tv} + 2t^2v' - \cancel{2t}v = 0$

$t^3v'' + 4t^2v' = 0$ $\quad u = v'$

$t^3 \frac{du}{dt} = -4t^2 u$ $\quad \frac{du}{dt} = v''$

$\frac{du}{dt} = \frac{-4}{t}u \rightarrow \int \frac{du}{u} = \int -\frac{4}{t}\,dt \rightarrow \ln u = -4\ln t + C$

$\cancel{\ln}\, u = \cancel{\ln}(At^{-4})$

$v = \int t^{-4}\,dt = \frac{t^{-3}}{-3} \cancel{+C} = -\frac{1}{3}t^{-3}$ $\quad u = At^{-4}$

or t^{-3}

$y_2 = tv = t(t^{-3}) = t^{-2}$

$y_1 = t,\ y_2 = t^{-2}$

b. $(x-1)y'' - xy' + y = 0$ $\quad y_1 = e^x$

$y_2 = ve^x$

$y_2' = v'e^x + ve^x$

$y_2'' = v''e^x + 2v'e^x + ve^x$

6b cont'd.

$$(x-1)(v''e^{x}+2v'e^{x}+ve^{x})-x(v'e^{x}+ve^{x})+ve^{x}=0$$

$\div e^{x}$

$$(x-1)(v''+2v'+v)-x(v'+v)+v=0$$

$$xv''+2xv'+\cancel{vx}-v''-2v'-\cancel{v}-xv'-\cancel{xv}+\cancel{v}=0$$

$$(x-1)v''+v'(x-2)=0 \qquad v'=u \quad u'=v''$$

$$(x-1)\frac{du}{dx}=-(x-2)u \rightarrow \frac{du}{u}=-\frac{(x-2)}{x-1}dx$$

$$x-1\overline{\big)-x+2}$$ with quotient -1, $\pm x \mp 1$, remainder 1

$$\int\frac{du}{u}=\int -1+\frac{1}{x-1}\,dx$$

$$\ln u=-x+\ln(x-1)+\cancel{C} \rightarrow \ln(e^{-x}(x-1))$$

$$u=e^{-x}(x-1)$$

$u=x-1 \quad dv=e^{-x}$

$du=dx \quad v=-e^{-x}$

$$v=\int e^{-x}(x-1)dx$$

$$=-(x-1)e^{-x}+\int e^{-x}dx = -(x-1)e^{-x}-e^{-x}+C$$

$$e^{-x}[-x+1-1]=-xe^{-x}+\cancel{C}$$

$$y_2=ve^{x}=(-xe^{-x})e^{x}=x$$

$$y_1=e^{x},\ y_2=x$$