MTH 291 Skills #7 Key

1a. $\begin{bmatrix} 1 & 6 \\ 2 & 5 \end{bmatrix}$ $(1-\lambda)(5-\lambda)-12=0$

$\lambda^2-6\lambda+5-12=0$

$\lambda^2-6\lambda-7=0$

$(\lambda-7)(\lambda+1)=0$

$\lambda=7, \lambda=-1$

$\lambda=-1$

$\begin{bmatrix} 2 & 6 \\ 2 & 6 \end{bmatrix}$ $2x_1+6x_2=0$

$x_1=-3x_2$ $\vec{v}_1=\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\lambda=7$

$\begin{bmatrix} -6 & 6 \\ 2 & -2 \end{bmatrix}$ $2x_1-2x_2=0$

$x_1=x_2$ $\vec{v}_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

b. $\begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}$ $(-1-\lambda)(1-\lambda)-3=0$

$\lambda^2-1-3=0$

$\lambda^2-4=0$

$(\lambda-2)(\lambda+2)=0$

$\lambda=\pm 2$

$\lambda=2$

$\begin{bmatrix} -3 & 1 \\ 3 & -1 \end{bmatrix}$ $3x_1-x_2=0$

$x_1=\frac{1}{3}x_2$ $\vec{v}_1=\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\lambda=-2$

$\begin{bmatrix} +1 & 1 \\ 3 & 3 \end{bmatrix}$ $x_1+x_2=0$

$x_1=-x_2$ $\vec{v}_2=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

c. $\begin{bmatrix} -4 & 1 \\ 6 & -5 \end{bmatrix}$ $(-4-\lambda)(-5-\lambda)-6=0$

$\lambda^2+9\lambda+20-6=0$

$\lambda^2+9\lambda+14=0$

$(\lambda+7)(\lambda+2)=0$

$\lambda=-7,-2$

$\lambda=-7$

$\begin{bmatrix} 3 & 1 \\ 6 & 2 \end{bmatrix}$ $3x_1+x_2=0$

$x_1=-\frac{1}{3}x_2$ $\vec{v}_1=\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\lambda=-2$

$\begin{bmatrix} -2 & 1 \\ 6 & -3 \end{bmatrix}$ $-2x_1+x_2=0$

$x_1=\frac{1}{2}x_2$ $\vec{v}_2=\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

d. $\begin{bmatrix} -2 & 2 \\ -5 & 6 \end{bmatrix}$ $(-2-\lambda)(6-\lambda)+10=0$

$\lambda^2-4\lambda-12+10=0$

$\lambda^2-4\lambda-2=0$

$\lambda=\frac{4\pm\sqrt{16+8}}{2}=\frac{4\pm2\sqrt{6}}{2}$

$=2\pm\sqrt{6}$

$\lambda=2+\sqrt{6}$

$\begin{bmatrix} -4-\sqrt{6} & 2 \\ -5 & 4-\sqrt{6} \end{bmatrix}$ $-5x_1+(4-\sqrt{6})x_2=0$

$x_1=\frac{4-\sqrt{6}}{5}x_2$

$\vec{v}_1=\begin{bmatrix} 4-\sqrt{6} \\ 5 \end{bmatrix}$ $\vec{v}_2=\begin{bmatrix} 4+\sqrt{6} \\ 5 \end{bmatrix}$

1e. $\begin{bmatrix} 2 & 9 \\ 1 & 10 \end{bmatrix}$ $(2-\lambda)(10-\lambda)-9=0$ $\lambda^2-12\lambda+20-9=0$ $\lambda^2-12\lambda+11=0$ $(\lambda-11)(\lambda-1)=0$ $\lambda=11,1$

$\lambda_1=11$

$\begin{bmatrix} -9 & 9 \\ 1 & -1 \end{bmatrix}$ $x_1-x_2=0$ $x_1=x_2$ $\vec{v}_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda_2=1$

$\begin{bmatrix} 1 & 9 \\ 1 & 9 \end{bmatrix}$ $x_1+9x_2=0$ $x_1=-9x_2$ $\vec{v}_2=\begin{bmatrix} -9 \\ 1 \end{bmatrix}$

f. $\begin{bmatrix} -2 & 5 \\ 7 & 0 \end{bmatrix}$ $(-2-\lambda)(-\lambda)-35=0$ $\lambda^2+2\lambda-35=0$ $(\lambda+7)(\lambda-5)=0$ $\lambda=-7,5$

$\lambda_1=-7$

$\begin{bmatrix} 5 & 5 \\ 7 & 7 \end{bmatrix}$ $5x_1+5x_2=0$ $x_1=-x_2$ $\vec{v}_1=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2=5$

$\begin{bmatrix} -7 & 5 \\ 7 & -5 \end{bmatrix}$ $7x_1-5x_2=0$ $x_1=\frac{5}{7}x_2$ $\vec{v}_2=\begin{bmatrix} 5 \\ 7 \end{bmatrix}$

g. $\begin{bmatrix} 3 & -2 \\ 2 & 3 \end{bmatrix}$ $(3-\lambda)(3-\lambda)+4=0$ $\lambda^2-6\lambda+9+4=0$ $\lambda^2-6\lambda+13=0$ $\lambda=\frac{6\pm\sqrt{36-52}}{2}=\frac{6\pm 4i}{2}=3\pm 2i$

$\lambda=3+2i$

$\begin{bmatrix} 3-(3+2i) & -2 \\ 2 & 3-(3+2i) \end{bmatrix}=\begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix}$ $2x_1-2ix_2=0$ $x_1=ix_2$ $\vec{v}_1=\begin{bmatrix} i \\ 1 \end{bmatrix}$ $\vec{v}_2=\begin{bmatrix} -i \\ 1 \end{bmatrix}$

h. $\begin{bmatrix} -2 & 5 & 3 \\ 0 & 2 & -4 \\ 0 & -1 & 2 \end{bmatrix}$ $(-2-\lambda)\left[(2-\lambda)(2-\lambda)-4\right]=0$ $(-2-\lambda)\left[\lambda^2-4\lambda+4-4\right]=0$ $(-2-\lambda)(\lambda^2-4\lambda)=0$ $\lambda=-2$ $\lambda=0$, $\lambda=4$

$\lambda=-2$

$\begin{bmatrix} 0 & 5 & 3 \\ 0 & 4 & -4 \\ 0 & -1 & 4 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ $\vec{v}_1=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\lambda_2=0$

$\begin{bmatrix} -2 & 5 & 3 \\ 0 & 2 & -4 \\ 0 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -13/2 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ $x_1=13/2\,x_3$ $x_2=2x_3$ $\vec{v}_2=\begin{bmatrix} 13 \\ 4 \\ 2 \end{bmatrix}$

11h cont'd

3

$\lambda = 4$

$$\begin{bmatrix} -6 & 5 & 3 \\ 0 & -2 & -4 \\ 0 & -1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 7/6 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 = -7/6\, x_3 \\ x_2 = -2x_3 \end{matrix} \quad \vec{v}_2 = \begin{bmatrix} -7 \\ -12 \\ 6 \end{bmatrix}$$

i. $\begin{bmatrix} 4 & 5 \\ 6 & 11 \end{bmatrix}$ $(4-\lambda)(11-\lambda) - 30 = 0$ $\lambda^2 - 15\lambda + 14 = 0$ $\lambda = 14, 1$

$\lambda^2 - 15\lambda + 44 - 30 = 0$ $(\lambda - 14)(\lambda - 1) = 0$

$\lambda_1 = 14$

$\begin{bmatrix} -10 & 5 \\ 6 & -3 \end{bmatrix}$ $6x_1 - 3x_2 = 0$ $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$x_1 = \frac{1}{2}x_2$

$\lambda_2 = 1$

$\begin{bmatrix} 3 & 5 \\ 6 & 10 \end{bmatrix}$ $3x_1 + 5x_2 = 0$ $\vec{v}_2 = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

$x_1 = -\frac{5}{3}x_2$

j. $\begin{bmatrix} -3 & 7 \\ 5 & -1 \end{bmatrix}$ $(-3-\lambda)(-1-\lambda) - 35 = 0$ $\lambda^2 + 4\lambda - 32 = 0$

$\lambda^2 + 4\lambda + 3 - 35 = 0$ $(\lambda + 8)(\lambda - 4) = 0$

$\lambda = -8, 4$

$\lambda = -8$

$\begin{bmatrix} 5 & 7 \\ 5 & 7 \end{bmatrix}$ $5x_1 + 7x_2 = 0$ $\vec{v}_1 = \begin{bmatrix} -7 \\ 5 \end{bmatrix}$

$x_1 = -\frac{7}{5}x_2$

$\lambda = 4$

$\begin{bmatrix} -7 & 7 \\ 5 & -5 \end{bmatrix}$ $5x_1 - 5x_2 = 0$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x_1 = x_2$

k. $\begin{bmatrix} -4 & 5 \\ -5 & -4 \end{bmatrix}$ $(-4-\lambda)(-4-\lambda) + 25 = 0$ $\lambda^2 + 8\lambda + 41 = 0$

$\lambda^2 + 8\lambda + 16 + 25 = 0$ $\lambda = \frac{-8 \pm \sqrt{64 - 164}}{2} = \frac{-8 \pm 10i}{2}$

$= -4 \pm 5i$

$\lambda = -4 + 5i$

$$\begin{bmatrix} -4-(-4+5i) & 5 \\ -5 & -4-(-4+5i) \end{bmatrix} = \begin{bmatrix} -5i & 5 \\ -5 & -5i \end{bmatrix}$$

$-5x_1 - 5ix_2 = 0$ $\vec{v}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$

$x_1 = -ix_2$

2\. a. $e^{1+2i} = e^1 e^{2i} = e\cos 2 + ie\sin 2$

b. $2^{1-i} = 2^1 2^{-i} = 2e^{-(\ln 2)i} = 2\cos(\ln 2) + i2\sin(-\ln 2)$

c. $e^{2-\pi/2 i} = e^2 e^{-\pi/2} = e^2\cos(-\pi/2) + e^2 i\sin(-\pi/2)$

$= e^2(0) - e^2 i = -e^2 i$

3a. $\vec{x}' = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix}\vec{x}$ $\quad (1-\lambda)(-4-\lambda)+6=0$ $\quad \lambda^2+3\lambda+2=0$

$\lambda^2+3\lambda-4+6=0$ $\quad (\lambda+2)(\lambda+1)=0$

$\lambda = -2, -1$

$\lambda_1 = -2$

$\begin{bmatrix} 3 & -2 \\ 3 & -2 \end{bmatrix}$ $\quad 3x_1 - 2x_2 = 0$ $\quad \vec{v}_1 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$x_1 = \frac{2}{3}x_2$

$\lambda = -1$

$\begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix}$ $\quad 3x_1 - 3x_2 = 0$ $\quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x_1 = x_2$ $\quad \vec{x} = c_1\begin{bmatrix} 2 \\ 3 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t}$

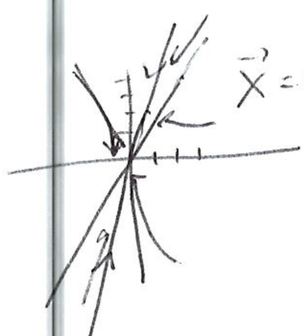
b. $t\vec{x}' = \begin{bmatrix} 4 & -3 \\ 8 & -6 \end{bmatrix}\vec{x}$ $\quad (4-\lambda)(-6-\lambda)+24=0$ $\quad \lambda^2+2\lambda=0$

$\lambda^2+2\lambda-24+24=0$ $\quad \lambda(\lambda+2)=0$ $\quad \lambda = 0, -2$

$\lambda = 0$

$4x_1 - 3x_2 = 0$ $\quad \vec{v}_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$x_1 = \frac{3}{4}x_2$



$\vec{x} = c_1\begin{bmatrix} 3 \\ 4 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 2 \end{bmatrix}t^{-2}$

$\lambda = -2$

$\begin{bmatrix} 6 & -3 \\ 8 & -4 \end{bmatrix}$ $\quad 8x_1 - 4x_2 = 0$ $\quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$x_1 = \frac{1}{2}x_2$

c. $\vec{x}' = \begin{bmatrix} -2 & 1 \\ -8 & 2 \end{bmatrix}\vec{x}$ $\quad (-2-\lambda)(2-\lambda)+8=0$ $\quad \lambda^2+4=0$

$\lambda^2-4+8=0$ $\quad \lambda = \pm 2i$

$\lambda_1 = 2i$

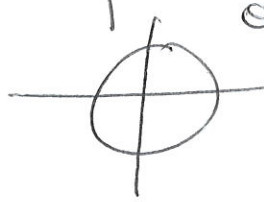
$\begin{bmatrix} -2-2i & 1 \\ -8 & 2-2i \end{bmatrix}$ $\quad -8x_1 + (2-2i)x_2 =$ $\quad \vec{v}_1 = \begin{bmatrix} 1-i \\ 4 \end{bmatrix}$ $\quad e^{2ti} = \cos 2t + i\sin 2t$

$x_1 = \frac{1-i}{4}x_2$

$\begin{bmatrix} 1-i \\ 4 \end{bmatrix}(\cos 2t + i\sin 2t) = \begin{bmatrix} \cos 2t + \sin 2t + i\sin 2t - i\cos 2t \\ 4\cos 2t + 4i\sin 2t \end{bmatrix}$

Stable. orbit

$\vec{x} = c_1\begin{bmatrix} \cos 2t + \sin 2t \\ 4\cos 2t \end{bmatrix} + c_2\begin{bmatrix} \sin 2t - \cos 2t \\ 4\sin 2t \end{bmatrix}$



⑤

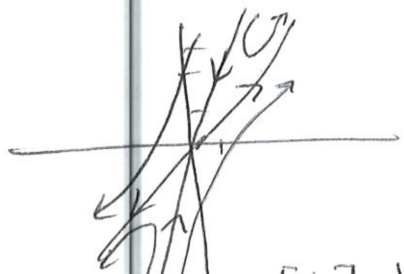
3d. $t\vec{x}' = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix}\vec{x}$ $\quad (2-\lambda)(-2-\lambda)+3=0$ $\quad \lambda^2-1=0$

$\lambda^2-4+3=0 \quad \lambda=\pm 1$

$\lambda_1=1$

$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$ $\quad x_1-x_2=0$ $\quad \vec{v}_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x_1=x_2$



$\lambda_2=-1$

$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$ $\quad 3x_1-x_2=0$ $\quad \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$3x_1=x_2$

$$\vec{x}=c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}t + c_2\begin{bmatrix} 1 \\ 3 \end{bmatrix}\frac{1}{t}$$

e. $\vec{x}' = \begin{bmatrix} 7 & -1 \\ 3 & 3 \end{bmatrix}\vec{x}$ $\quad (7-\lambda)(3-\lambda)+3=0$ $\quad \lambda^2-10\lambda+24=0$

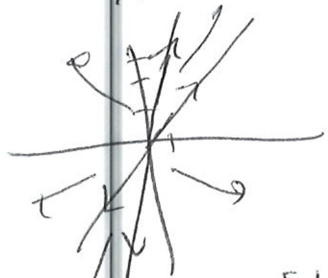
$\lambda^2-10\lambda+21+3=0 \quad (\lambda-6)(\lambda-4)=0$

$\lambda=6,4$

$\lambda_1=6$

$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$ $\quad x_1-x_2=0$ $\quad \vec{v}_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x_1=x_2$



$\lambda_2=4$

$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$ $\quad 3x_1-x_2=0$ $\quad \vec{v}_2=\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$3x_1=x_2$

$$\vec{x}=c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{6t} + c_2\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{4t}$$

f. $\vec{x}' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix}\vec{x}$ $\quad (3-\lambda)(-1-\lambda)+8=0$ $\quad \lambda^2-2\lambda+5=0$

$\lambda^2-2\lambda-3+8=0 \quad \lambda=\frac{2\pm\sqrt{4-20}}{2}=\frac{2\pm 4i}{2}=1\pm 2i$

$\begin{bmatrix} 3-(1+2i) & -2 \\ 4 & -1-(1+2i) \end{bmatrix} = \begin{bmatrix} 2-2i & -2 \\ 4 & -2-2i \end{bmatrix}$ $\quad 4x_1-(2+2i)x_2=0$ $\quad \vec{v}_1=\begin{bmatrix} 1+i \\ 2 \end{bmatrix}$

$x_1=\frac{1+i}{2}x_2$

$$e^t\begin{bmatrix} 1+i \\ 2 \end{bmatrix}(\cos 2t + i\sin 2t) = e^t\begin{bmatrix} \cos 2t + i\cos 2t + i\sin 2t - \sin 2t \\ 2\cos 2t + 2i\sin 2t \end{bmatrix}$$

$$\vec{x}=c_1\begin{bmatrix} \cos 2t - \sin 2t \\ 2\cos 2t \end{bmatrix}e^t + c_2\begin{bmatrix} \cos 2t + \sin 2t \\ 2\sin 2t \end{bmatrix}e^t$$

g. $\vec{x}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\vec{x}$ $\quad (-2-\lambda)(-2-\lambda)-1=0$

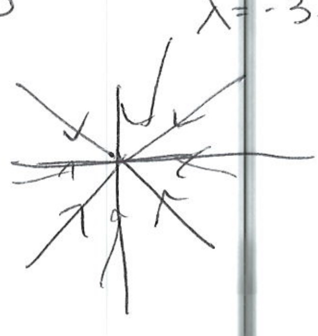
$\lambda^2+4\lambda+4-1=0 \quad (\lambda+3)(\lambda+1)=0$

$\lambda^2+4\lambda+3=0 \quad \lambda=-3,-1$

$\lambda=-3$

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad x_1+x_2=0$ $\quad \vec{v}_1=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$x_1=-x_2$



$$\vec{x}=c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}e^{-3t} + c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{-t}$$

$\lambda=-1$

$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ $\quad x_1-x_2=0$ $\quad \vec{v}_2=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x_1=x_2$

3h. $\vec{x}' = \begin{bmatrix}4 & -3\\ 6 & -2\end{bmatrix}\vec{x}$ $\quad (4-\lambda)(-2-\lambda)+18=0 \quad \lambda^2-2\lambda+10=0$

$\lambda^2-2\lambda-8+18=0 \quad \lambda=\frac{2\pm\sqrt{4-40}}{2}=\frac{2\pm 6i}{2}=1+3i$

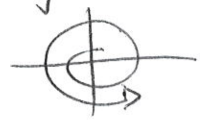
$\lambda_1 = 1+3i$

$\begin{bmatrix}4-(1+3i) & -3\\ 6 & -2(1+3i)\end{bmatrix} = \begin{bmatrix}3-3i & -3\\ 6 & -3-3i\end{bmatrix} \quad 6x_1-(3+3i)x_2=0 \quad \vec{v}_1=\begin{bmatrix}1+i\\ 2\end{bmatrix}$

$x_1 = \frac{1+i}{2}x_2$

$e^t\begin{bmatrix}1+i\\ 2\end{bmatrix}(\cos 3t + i\sin 3t) = e^t\begin{bmatrix}\cos 3t + i\sin 3t + i\cos 3t - \sin 3t\\ 2\cos 3t + 2i\sin 3t\end{bmatrix}$

$\vec{x} = c_1\begin{bmatrix}\cos 3t - \sin 3t\\ 2\cos 3t\end{bmatrix}e^t + c_2\begin{bmatrix}\sin 3t + \cos 3t\\ 2\sin 3t\end{bmatrix}e^t$



i. $\vec{x}' = \begin{bmatrix}1 & -5\\ 1 & -3\end{bmatrix}\vec{x}$ $\quad (1-\lambda)(-3-\lambda)+5=0 \quad \lambda^2+2\lambda+2=0$

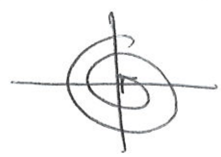
$\lambda^2+2\lambda-3+5=0 \quad \lambda=\frac{-2\pm\sqrt{4-8}}{2}=\frac{-2\pm 2i}{2}=-1\pm i$

$\begin{bmatrix}1-(-1+i) & -5\\ 1 & -3-(-1+i)\end{bmatrix} = \begin{bmatrix}2-i & -5\\ 1 & -2-i\end{bmatrix} \quad 1x_1-(2+i)x_2=0 \quad \vec{v}_1=\begin{bmatrix}2+i\\ 1\end{bmatrix}$

$x_1=(2+i)x_2$

$e^{-t}\begin{bmatrix}2+i\\ 1\end{bmatrix}(\cos t + i\sin t) = e^{-t}\begin{bmatrix}2\cos t + 2i\sin t + i\cos t - \sin t\\ \cos t + i\sin t\end{bmatrix}$

$\vec{x} = c_1e^{-t}\begin{bmatrix}2\cos t - \sin t\\ \cos t\end{bmatrix} + c_2e^{-t}\begin{bmatrix}2\sin t + \cos t\\ \sin t\end{bmatrix}$



4.a. $\frac{d^2x_1}{dt^2} = -\frac{(2+3)}{1}x_1 + \frac{3}{1}x_2 \rightarrow \frac{d^2x_1}{dt^2} = -5x_1 + 3x_2$

$\frac{d^2x_2}{dt^2} = +\frac{3}{5}x_1 - \frac{(3)}{5}x_2 \rightarrow \frac{d^2x_2}{dt^2} = \frac{3}{5}x_1 - \frac{3}{5}x_2$

b. $\frac{d^2x_1}{dt^2} = -\frac{(1+4)}{1}x_1 + 4x_2 \rightarrow \frac{d^2x_1}{dt^2} = -5x_1 + 4x_2$

$\frac{d^2x_2}{dt^2} = -\frac{(4+1)}{1}x_2 + 1x_1 \rightarrow \frac{d^2x_2}{dt^2} = x_1 - 5x_2$

c. $\frac{d^2x_1}{dt^2} = -\frac{(4+1)}{1}x_1 + 1x_2 \rightarrow \frac{d^2x_1}{dt^2} = -5x_1 + x_2$

$\frac{d^2x_2}{dt^2} = 1x_2 - x_1 \qquad \frac{d^2x_2}{dt^2} = -x_1 + x_2$

4d. $\frac{d^2x_1}{dt^2} = -\frac{(1+1)}{1}x_1 + \frac{1x_2}{1} \rightarrow \frac{d^2x_1}{dt^2} = -2x_1 + x_2$

$\frac{d^2x_2}{dt^2} = -\frac{(1+1)}{2}x_2 + \frac{1x_1}{2} + \frac{1x_3}{2}$ $\quad \frac{d^2x_2}{dt^2} = \frac{x_1}{2} - x_2 + \frac{1}{2}x_3$

$\frac{d^2x_3}{dt^2} = -\frac{(1)}{3}x_3 + \frac{1x_2}{3}$ $\quad \frac{d^2x_3}{dt^2} = \frac{x_2}{3} - \frac{1}{3}x_3$

5a. $\vec{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\vec{x}$ $\quad (x(0)) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$(5-\lambda)(1-\lambda)+3=0$ $\quad (\lambda-4)(\lambda-2)=0$

$\lambda^2-6\lambda+5+3=0$ $\quad \lambda=4,2$

$\lambda^2-6\lambda+8=0$

$\lambda_1=4$

$\begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix}$ $\quad x_1-x_2=0$ $\quad \vec{v}_1=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x_1=x_2$

$\lambda_2=2$

$\begin{bmatrix} 3 & -1 \\ 3 & -1 \end{bmatrix}$ $\quad 3x_1-x_2=0$ $\quad \vec{v}_2=\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\quad \vec{x}=c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{4t}+c_2\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{2t}$

$x_1=\frac{1}{3}x_2$

$c_1+c_2=2$ $\quad c_1=7/2$

$c_1+3c_2=-1$ $\quad c_2=-3/2$

$\vec{x}=\frac{7}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{4t}-3/2\begin{bmatrix} 1 \\ 3 \end{bmatrix}e^{2t}$

b. $\vec{x}'=\begin{pmatrix} 0 & 0 & -1 \\ 2 & 0 & 0 \\ -1 & 2 & 4 \end{pmatrix}\vec{x}$, $\quad \vec{x}(0)=\begin{pmatrix} 7 \\ 5 \\ 5 \end{pmatrix}$

$(-\lambda)[(-\lambda)(4-\lambda)]-0+(-1)[(2)(2)-\lambda]$

$-\lambda[\lambda^2-4\lambda]-(4-\lambda)$

$-\lambda(\lambda)(\lambda-4)+(\lambda-4)$

$(\lambda-4)(-\lambda^2+1)$ $\quad \lambda=4,\pm1$

$\lambda_1=4$

$\begin{pmatrix} -4 & 0 & -1 \\ 2 & -4 & 0 \\ -1 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1/4 \\ 0 & 1 & 1/8 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x_1+\frac{1}{4}x_3=0$ $\quad \vec{v}_1=\begin{bmatrix} -2 \\ -1 \\ 8 \end{bmatrix}$

$x_2+\frac{1}{8}x_3=0$

$\lambda_2=1$

$\begin{pmatrix} -1 & 0 & -1 \\ 2 & -1 & 0 \\ -1 & 2 & 3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x_1=-x_3$ $\quad \vec{v}_2=\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

$x_2=-2x_3$

$\lambda_3=-1$

$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ -1 & 2 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad x_1=x_3$ $\quad \vec{v}_3=\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

$x_2=-2x_3$

5b cont'd

$$\vec{X} = c_1 \begin{bmatrix} -2 \\ -1 \\ 8 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-t}$$

$-2c_1 - c_2 + c_3 = 7$ $\qquad c_1 = 1$

$-c_1 - 2c_2 - 2c_3 = 5$ $\qquad c_2 = -6$

$8c_1 + c_2 + c_3 = 5$ $\qquad c_3 = 3$

$$\vec{X} = \begin{bmatrix} -2 \\ -1 \\ 8 \end{bmatrix} e^{4t} - 6 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} e^{t} + 3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{-t}$$

6a. $\vec{X}' = \begin{pmatrix} 3 & -2 \\ 4 & 1 \end{pmatrix} \vec{X}$ $\qquad (3-\lambda)(1-\lambda) + 8 = 0$ $\qquad \lambda^2 - 4\lambda + 12 = 0$

$\lambda^2 - 4\lambda + 4 + 8 = 0$ $\qquad \lambda = \frac{4 \pm \sqrt{4-48}}{2} = \frac{4 \pm 2\sqrt{11}i}{2} = 2 \pm \sqrt{11}i$

$\lambda = 2 + \sqrt{11}i$

$$\begin{pmatrix} 3-(2+\sqrt{11}i) & -2 \\ 4 & 1-(2+\sqrt{11}i) \end{pmatrix} = \begin{pmatrix} 1-\sqrt{11}i & -2 \\ 4 & -1-\sqrt{11}i \end{pmatrix}$$

$4x_1 - (1+\sqrt{11}i)x_2 = 0$ $\qquad \vec{v}_1 = \begin{bmatrix} 1+\sqrt{11}i \\ 4 \end{bmatrix}$

$x_1 = \frac{1+\sqrt{11}i}{4} x_2$

$$e^{2t} \begin{bmatrix} 1+\sqrt{11}i \\ 4 \end{bmatrix} (\cos\sqrt{11}t + i\sin\sqrt{11}t) = e^{2t} \begin{bmatrix} \cos\sqrt{11}t + i\sin\sqrt{11}t + \sqrt{11}i\cos\sqrt{11}t - \sqrt{11}\sin\sqrt{11}t \\ 4\cos\sqrt{11}t + 4i\sin\sqrt{11}t \end{bmatrix}$$

$$\vec{X} = c_1 \begin{bmatrix} \cos\sqrt{11}t - \sqrt{11}\sin\sqrt{11}t \\ 4\cos\sqrt{11}t \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} \sin\sqrt{11}t + \sqrt{11}\cos\sqrt{11}t \\ 4\sin\sqrt{11}t \end{bmatrix} e^{2t}$$

as $t \to \infty$ the system spirals outward to ∞

b. $\vec{X}' = \begin{pmatrix} 1 & 2 \\ -5 & -1 \end{pmatrix} \vec{X}$ $\qquad (1-\lambda)(-1-\lambda) + 10 = 0$ $\qquad \lambda^2 + 9 = 0$

$\lambda^2 - 1 + 10 = 0$ $\qquad \lambda = \pm 3i$

$\lambda_1 = 3i$

$\begin{bmatrix} 1-3i & 2 \\ -5 & -1-3i \end{bmatrix}$ $\qquad -5x_1 - (1+3i)x_2 = 0$ $\qquad \vec{V}_1 = \begin{bmatrix} 1+3i \\ -5 \end{bmatrix}$

$x_1 = \frac{1+3i}{-5} x_2$

$$\begin{bmatrix} 1+3i \\ -5 \end{bmatrix} (\cos 3t + i\sin 3t) = \begin{bmatrix} \cos 3t + i\sin 3t + 3i\cos 3t - 3\sin 3t \\ -5\cos 3t - 5i\sin 3t \end{bmatrix}$$

$$\vec{X} = c_1 \begin{bmatrix} \cos 3t - 3\sin 3t \\ -5\cos 3t \end{bmatrix} + c_2 \begin{bmatrix} \sin 3t + 3\cos 3t \\ -5\sin 3t \end{bmatrix}$$

as $t \to \infty$, remains in stable orbit

7a. $\vec{X} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{X}$ $\qquad (3-\lambda)(-2-\lambda) + 4 = 0$ $\qquad \lambda^2 - \lambda - 2 = 0$

$\lambda^2 - \lambda - 6 + 4 = 0$ $\qquad (\lambda - 2)(\lambda + 1) = 0$

$\lambda = 2, -1$

7a contd

$\lambda_1 = 2 \quad \begin{bmatrix} 1 & -2 \\ 2 & \cancel{-4} \end{bmatrix}$. $\quad x_1 - 2x_2 = 0$, $x_1 = 2x_2$ $\quad \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad \lambda_2 = -1 \quad \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}$ $\quad 2x_1 - x_2 = 0$, $x_1 = \frac{1}{2}x_2$ $\quad \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$\vec{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$$

7b. $\vec{x} = \begin{pmatrix} 2 & 2+i \\ -1 & -1-i \end{pmatrix} \vec{x}$

$\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} -2-i \\ 1 \end{bmatrix}$

$\lambda_1 = -i \qquad \lambda_2 = 1$

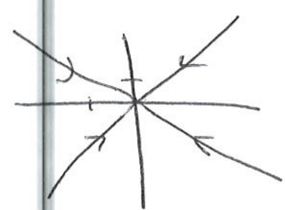
$$\vec{x} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} (\cos t - i \sin t) + c_2 \begin{bmatrix} -2-i \\ 1 \end{bmatrix} e^t$$

Since the matrix is complex, the solution is complex

7c. $\vec{x}\,' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x}$ $\qquad (-2-\lambda)(-2-\lambda) - 1 = 0$, $\lambda^2 + 4\lambda + 4 - 1 = 0$ $\qquad \lambda^2 + 4\lambda + 3 = 0$, $(\lambda+3)(\lambda+1) = 0$, $\lambda = -3, -1$

$\lambda_1 = -3 \quad \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad x_1 + x_2 = 0$, $x_1 = -x_2$ $\quad \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad \lambda_1 = -1 \quad \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$ $\quad x_1 - x_2 = 0$, $x_1 = x_2$ $\quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\vec{x} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

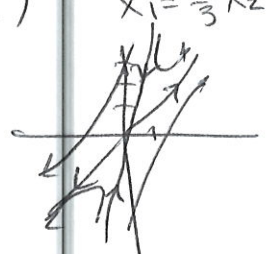


as time goes to ∞, the system goes to 0

d. $t\vec{x}\,' = \begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} \vec{x}$ $\qquad (2-\lambda)(-2-\lambda) + 3 = 0$, $\lambda^2 - 4 + 3 = 0$, $\lambda^2 - 1 = 0$ $\qquad \lambda = \pm 1$

$\lambda_1 = 1 \quad \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$ $\quad x_1 - x_2 = 0$, $x_1 = x_2$ $\quad \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \lambda_2 = -1 \quad \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}$ $\quad 3x_1 - x_2 = 0$, $x_1 = \frac{1}{3}x_2$ $\quad \vec{v}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$$



as $t \to \infty$, system approaches $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ vector heading to ∞.