MTH 291 Skills #8 Key

1a. $6y'' - 5y' + y = 0$

$6r^2 - 5r + 1 = 0 \qquad (3r-1)(2r-1) = 0$

$r = 1/3,\ r = 1/2$

$y = c_1 e^{1/3 t} + c_2 e^{1/2 t} \rightarrow 4 = c_1 + c_2$

$y' = 1/3 c_1 e^{1/3 t} + 1/2 c_2 e^{1/2 t} \rightarrow 0 = 1/3 c_1 + 1/2 c_2$

$c_1 = 12,\ c_2 = -8$

$y = 12e^{1/3 t} - 8e^{1/2 t} \qquad$ as $t \to \infty,\ y \to -\infty$

max at $t = 0. \qquad (0,4)$

b. $2y'' - 3y' + y = 0 \qquad (2r-1)(r-1) = 0$

$2r^2 - 3r + 1 = 0 \qquad r = 1/2,\ r = 1$

$y = c_1 e^{1/2 t} + c_2 e^{t} \rightarrow 2 = c_1 + c_2$

$y' = 1/2 c_1 e^{1/2 t} + c_2 e^{t} \rightarrow 1/2 = 1/2 c_1 + c_2$

$c_1 = 3,\ c_2 = -1$

$y = 3e^{1/2 t} - e^{t} \qquad$ as $t \to \infty, y \to -\infty$

max at $\approx t = .810931,\ y = 2.25$

c. $y'' + 4y' + 5y = 0$

$r^2 + 4r + 5 = 0 \qquad r = \dfrac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$

$e^{(-2+i)t} = e^{-2t}(\cos t + i \sin t) \rightarrow$

$y = c_1 e^{-2t} \cos t + c_2 e^{-2t} \sin t \qquad 1 = c_1$

$y' = -2e^{-2t}\cos t - e^{-2t}\cancel{\sin t}^{0} - 2c_2 e^{-2t}\cancel{\sin t}^{0} + c_2 e^{-2t}\cos t = 0$

$-2\cancel{e^{-2t}}\cancel{\cos t} + c_2 \cancel{e^{-2t}}\cancel{\cos t} = 0 \Rightarrow c_2 = 2$

$y = e^{-2t}\cos t + 2e^{-2t}\sin t \qquad$ as $t \to \infty,\ y \to 0$

∞ # of critical points

(2)

1d. $y'' + 4y' + 4y = 0$ $\quad (r+2)^2 = 0$

$r^2 + 4r + 4 = 0$ $\quad r = -2$

$y = c_1 e^{-2t} + c_2 t e^{-2t}$ $\quad c_1 e^{+2} + c_2(-1)e^2 = 2$

$c_1 + c_2 = \frac{2}{e^2}$

$y' = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$

$-2c_1 e^2 + c_2 e^2 + 2c_2 e^2 = 1$

$-2c_1 e^2 + 3c_2 e^2 = 1$

$-2c_1 + 3c_2 = \frac{1}{e^2}$

$c_1 = c_2 \approx .13533528$

$y = .135 e^{-2t} + .135 t e^{-2t}$ $\quad$ as $t \to \infty$, $y \to 0$

max at $\approx -1/2 = t$, $y = .18348$

2a. $W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = \cos^2 t + \sin^2 t = 1 \neq 0$

Yes, this forms a fundamental set

b. $W = \begin{vmatrix} t^2 & 2t\ln t + t & t^{-4} \\ 2t & 2\ln t + 3 & -4t^{-5} \\ 2 & \frac{2}{t} & 20t^{-6} \end{vmatrix} = t^2\left[(2\ln t + 3)20t^{-6} + 8t^{-6}\right] + -$

$2t\left[(2t\ln t + t)20t^{-6} - \frac{2}{t^5}\right] +$

$2\left[(2t\ln t + t)(-4t^{-5}) - (2\ln t + 3)t^{-4}\right]$

$= t^2\left[\frac{40\ln t}{t^6} + \frac{60}{t^6} + \frac{8}{t^6}\right] - 2t\left[\frac{40\ln t}{t^5} + \frac{20}{t^5} - \frac{2}{t^5}\right] +$

$2\left[-\frac{8\ln t}{t^4} - \frac{4}{t^4} - \frac{2\ln t}{t^4} - \frac{3}{t^4}\right] =$

$\frac{40\ln t}{t^4} + \frac{68}{t^4} - \frac{80\ln t}{t^4} + \frac{18}{t^4} - \frac{20\ln t}{t^4} - \frac{14}{t^4} = \frac{-60\ln t + 72}{t^4}$

$\neq 0$

fundamental set

③

2c. $W = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = xe^x + x^2e^x - xe^x = x^2e^x \neq 0$

fundamental set

d. $W = \begin{vmatrix} \sinh t & \cosh t & e^t \\ \cosh t & \sinh t & e^t \\ \sinh t & \cosh t & e^t \end{vmatrix} = e^t(\sinh^2 t - \cosh^2 t) - e^t(\sinh t \cosh t - \sinh t \cosh t) + e^t(\cosh^2 t - \sinh^2 t) = 0$

not a fundamental set.

e. $W = \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t \cos t + e^t \sin t & e^t \cos t - e^t \sin t \end{vmatrix} =$

$\cancel{e^{2t} \sin t \cos t} - e^{2t}\sin^2 t - e^{2t}\cos^2 t - \cancel{e^{2t} \sin t \cos t} = -e^{2t}(\cos^2 t + \sin^2 t) =$

$-e^{2t} \neq 0$

fundamental set.

3a. $ty'' + 3y = t \quad y(1) = 1,\ y'(1) = 2$

$y'' + \frac{3}{t}y' = 1 \qquad W = e^{-\int \frac{3}{t}dt} = e^{-3\ln t} = t^{-3} \qquad (0, \infty)$

b. $t(t-4)y'' - 3ty' + 4y = 2,\ y(3) = 0,\ y'(3) = -1$

$y'' - \frac{3}{t-4}y' + \frac{4}{t(t-4)}y = \frac{2}{t(t-4)} \qquad W = e^{-\int \frac{3}{t-4}dt} = e^{-3\ln(t-4)} = \frac{1}{(t-4)^3} \qquad (-\infty, 4)$

c. $x^2(x^2-9)y'' - xy' + y = 0 \quad y(\pi/2) = 1,\ y'(\frac{\pi}{2}) = 0$

$y'' - \frac{\cancel{x}1}{x^{\cancel{2}}(x^2-9)}y' + \frac{y}{x^2(x^2-9)} = 0 \qquad W = e^{-\int \frac{1}{x(x^2-9)}dx} =$

$e^{-\frac{1}{18}(\ln(9-x^2) - 2\ln x)} = e^{-\frac{1}{18}\ln\left(\frac{9-x^2}{x^2}\right)} = \left(\frac{9-x^2}{x^2}\right)^{-1/18} = \left(\frac{x^2}{9-x^2}\right)^{1/18}$

4a. $n(n-1) + n + 1 = n^2 - n + n + 1 = 0 \Rightarrow n^2 + 1 = 0 \qquad n = \pm i$

$y = x^i \Rightarrow y = c_1\cos(\ln x) + c_2\sin(\ln x) \qquad x^i = e^{\ln x\, i} = \cos(\ln x) + i\sin(\ln x)$

b. $n(n-1) + 5n + 13 = n^2 - n + 5n + 13 = n^2 + 4n + 13 = 0$

$\frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm 6i}{2} = -2 \pm 3i \qquad t^{(-2+3i)} = t^{-2}t^{3i} = t^{-2}e^{3\ln t\, i} = t^{-2}(\cos(3\ln t) + i\sin(3\ln t))$

$y = c_1 t^{-2}\cos(3\ln t) + c_2 t^{-2}\sin(3\ln t)$

4c. $n(n-1)-n+5=0 \Rightarrow n^2-n-n+5 = n^2-2n+5=0$

$n=\frac{2\pm\sqrt{4-20}}{2} = \frac{2\pm 4i}{2} = 1\pm 2i \qquad t^{(1+2i)} = t\cdot t^{2i} = te^{2\ln t\, i}$

$= t(\cos(2\ln t) + i\sin(2\ln t))$

$y = c_1 t\cos(2\ln t) + c_2 t \sin(2\ln t)$

5a. $r^2+2r+2=0 \qquad r = \frac{-2\pm\sqrt{4-8}}{2} = \frac{-2\pm 2i}{2} = -1\pm i$

$y = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$

$Y(t) = Ae^{-t} + (Bt^2+Ct+D)(e^{-t}\cos t)t + (Et^2+Ft+G)(e^{-t}\sin t)t$

b. $r^2+4=0 \quad r=\pm 2i \quad y = c_1\cos 2t + c_2\sin 2t$

$Y(t) = (At^2+Bt+C)t\sin 2t + (Dt^2+Et+F)t\cos 2t$

6a. $r^2+1=0 \quad r=\pm i \quad y=\cos t, \sin t \quad W=1 = \begin{vmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{vmatrix} = 1$

$Y(t) = -\sin t\int \frac{\cos t\cdot\tan t}{1}\,dt + \cos t\int\frac{\sin t\tan t}{1}\,dt$

$= -\sin t\int\frac{\sin t}{\cos^2 t}\,dt + \cos t\int\frac{\sin^2 t}{\cos t}\,dt$

$= -\sin t(\sec t) + \cos t\left(-\sin t - \ln\left[\frac{\cos t/2 - \sin t/2}{\cos t/2 + \sin t/2}\right]\right)$

$= -\tan t + -\sin t\cos t + \cos\ln\left[\frac{\cos(t/2)-\sin(t/2)}{\cos t/2 + \sin(t/2)}\right]$

b. $W = \begin{vmatrix}1+t & e^t\\ 1 & e^t\end{vmatrix} = e^t(1+t) - e^t = te^t$

$Y(t) = (1+t)\int\frac{\cancel{e^t}\cdot t^2 e^{2t}}{\cancel{te^t}}\,dt - e^t\int\frac{(1+t)t^2 e^{2t}}{\cancel{te^t}}\,dt$

$= (1+t)\int te^{2t}\,dt - e^t\int te^t + t^2e^t\,dt =$

$=(1+t)\left(\frac{1}{4}e^{2t}(2t-1)\right) - e^t(te^t - e^t + t^2e^t - 2te^t + 2e^t)$

$= \frac{1}{4}e^{2t}(2t-1) + \frac{1}{4}te^{2t}(2t-1) - e^t(te^t - e^t + t^2e^t - 2te^t + 2e^t)$

$\frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + \frac{1}{2}t^2e^{2t} - \frac{1}{4}te^{2t} - te^{2t} + e^{2t} - t^2e^{2t} + 2te^{2t} - 2e^{2t}$

$= e^{2t}(-\frac{1}{2}t^2 + \frac{5}{4}t - \frac{5}{4})$

6c. $W = \begin{vmatrix} x^2 & x^2\ln x \\ 2x & 2x\ln x + x \end{vmatrix} = \cancel{2x^3\ln x} + x^3 - \cancel{2x^3\ln x} = x^3$

$$Y(t) = x^2\int \frac{\cancel{x^2}\ln x \cdot \cancel{x^2}\ln x}{\cancel{x^3}}\,dx - x^2\ln x\int \frac{\cancel{x^2}\ln x \cdot \cancel{x^2}}{\cancel{x^3}}\,dx$$

$$= x^2\int x(\ln x)^2\,dx - x^2\ln x\int x\ln x\,dx =$$

$$= x^2\left(\tfrac{1}{4}x^2(2\ln^2 x - 2\ln x + 1)\right) - x^2\ln x\left(\tfrac{1}{4}x^2(2\ln x - 1)\right)$$

$$= \tfrac{1}{4}x^4(2\ln^2 x - 2\ln x + 1) - \tfrac{1}{4}x^4\ln x(2\ln x - 1)$$

$$= +\cancel{\tfrac{1}{2}x^4\ln^2 x} - \tfrac{1}{2}x^4\ln x + \tfrac{1}{4}x^4 - \cancel{\tfrac{1}{2}x^4\ln^2 x} + \tfrac{1}{4}x^4\ln x$$

$$= -\tfrac{1}{4}x^4\ln x + \tfrac{1}{4}x^4$$

d. $y(t) = e^t,\ te^t$ $\quad W = \begin{vmatrix} e^t & te^t \\ e^t & e^t + te^t \end{vmatrix} = e^{2t} + \cancel{te^{2t}} - \cancel{te^{2t}} = e^{2t}$

$$Y(t) = e^t\int \frac{t\cancel{e^t}}{\cancel{e^{2t}}}\cdot\frac{\cancel{e^t}}{1+t^2}\,dt - te^t\int\frac{\cancel{e^t}}{\cancel{e^{2t}}}\cdot\frac{\cancel{e^t}}{1+t^2}\,dt$$

$$= e^t\int\frac{t}{1+t^2}\,dt - te^t\int\frac{1}{1+t^2}\,dt = \tfrac{1}{2}e^t\ln|1+t^2| - te^t\arctan t$$

7. a. $(\alpha-\lambda)(\alpha-\lambda) + 1 = 0$ $\qquad \lambda^2 - 2\alpha\lambda + (\alpha^2+1) = 0$

$\alpha^2 - 2\alpha\lambda + \lambda^2 + 1 = 0$ $\qquad \dfrac{+2\alpha \pm \sqrt{4\alpha^2 - 4(\alpha^2+1)}}{2}$

changes as discriminant goes through 0 $\rightarrow$ $4\alpha^2 - 4(\alpha^2+1) = 0$

$\cancel{4\alpha^2} - \cancel{4\alpha^2} - 4 = 0$ never 0

$\lambda = \alpha \pm 2i$

changes when real + or neg. $\quad \alpha = 0$

Use technology to draw phase portraits

b. $(2-\lambda)(-2-\lambda) + 5\alpha = 0$ $\quad \lambda^2 - 4 + 5\alpha = 0$ $\quad \lambda = \pm\sqrt{4-5\alpha}$

changes when $4 = 5\alpha$ $\quad \alpha = \frac{4}{5}$

8. a $X_2 = \dfrac{3x_1 - x_1'}{2}$ $\qquad x_2' = \dfrac{3x_1' - x_1''}{2}$ $\qquad x_1(0) = 3$

$x_1'(0) = 3(3) - 2(1/2)$
$= 9 - 1 = 8.$

$\frac{3}{2}X_1' - \frac{1}{2}X_1'' = 2x_1 - \cancel{2}\left(\frac{3x_1 - x_1'}{\cancel{2}}\right)$

$\frac{3}{2}x_1' - \frac{1}{2}x_1'' = 2x_1 - 3x_1 + x_1'$

$\frac{3}{2}x_1' - \frac{1}{2}x_1'' = -x_1 + x_1'$

$-\frac{1}{2}x_1'' + \frac{1}{2}x_1' + x_1 = 0 \Rightarrow x_1'' - x_1' - 2x_1 = 0$

b. $x_1' = 2x_2 \rightarrow x_1'' = 2x_2' \rightarrow \frac{1}{2}x_1'' = x_2'$

$x_2' = -2x_1 \rightarrow \frac{1}{2}x_1'' = -2x_1 \Rightarrow x_1'' = -4x_1 \Rightarrow x_1'' + 4x_1 = 0$

$x_1(0) = 3$
$x_1'(0) = 8$

9. $y'' + 2y' + 10y = 0$ $\qquad r = \dfrac{-2 \pm \sqrt{4-40}}{2} = \dfrac{-2 \pm 6i}{2} = -1 \pm 3i$

$r^2 + 2r + 10 = 0$

$y(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t + \frac{4}{10}$

Steady state solution is $\frac{4}{10}$ no resonance

$y(t) = A\cos 2t + B\sin 2t$

$y'(t) = -2A\sin 2t + 2B\cos 2t$

$y''(t) = -4A\cos 2t - 4B\sin 2t$

$-4A\cos 2t - 4B\sin 2t + 2(-2A\sin 2t + 2B\cos 2t) + 10(A\cos 2t + B\sin 2t)$

$= -4A\cos 2t - 4B\sin 2t - 4A\sin 2t + 4B\cos 2t + 10A\cos 2t + 10B\sin 2t$

$= (-4A + 10A + 4B)\cos 2t + (-4B + 10B - 4A)\sin 2t = 4\cos 2t$

$6A + 4B = 4 \qquad A = 6/13$
$-4A + 6B = 0 \qquad B = 4/13$

$y(t) = c_1 e^{-t}\cos 3t + c_2 e^{-t}\sin 3t + 6/13\cos 2t + 4/13\sin 2t$

no resonance