MTH 266, Exam #1, Part I, Fall 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available. On this portion of the exam, you may **NOT** use a calculator.

1. Given the system of equations $\begin{cases} x_1 + 5x_2 + 5x_3 = -9 \\ -x_1 - 2x_2 - x_3 = 4 \\ \quad -3x_2 - 6x_3 = 3 \end{cases}$, write the *system* as:

   a. An augmented matrix (3 points)

$$\left[\begin{array}{ccc|c} 1 & 5 & 5 & -9 \\ -1 & -2 & -1 & 4 \\ 0 & -3 & -6 & 3 \end{array}\right]$$

   b. A vector equation (3 points)

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 5 \\ -2 \\ -3 \end{bmatrix} x_2 + \begin{bmatrix} 5 \\ -1 \\ -6 \end{bmatrix} x_3 = \begin{bmatrix} -9 \\ 4 \\ 3 \end{bmatrix}$$

   c. A matrix equation. (3 points)

$$\begin{bmatrix} 1 & 5 & 5 \\ -1 & -2 & -1 \\ 0 & -3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -9 \\ 4 \\ 3 \end{bmatrix}$$

   d. Solve the system using the augmented matrix and row operations. State whether the solution of the system is consistent or inconsistent. If the system is consistent, state whether it is independent or dependent. Write an independent solution in vector form; write a dependent solution in parametric form. (7 points)

$R_1 + R_2 \to R_2$ $\left[\begin{array}{ccc|c} 1 & 5 & 5 & -9 \\ 0 & 3 & 4 & -5 \\ 0 & -3 & -6 & 3 \end{array}\right]$ $R_2 \leftrightarrow R_3$, $R_2 + R_3 \to R_3$ $\left[\begin{array}{ccc|c} 1 & 5 & 5 & -9 \\ 0 & -3 & -6 & 3 \\ 0 & 0 & -2 & -2 \end{array}\right]$ $-\frac{1}{3}R_2 \to R_2$, $-\frac{1}{2}R_3 \to R_3$

$\left[\begin{array}{ccc|c} 1 & 5 & 5 & -9 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right]$ $-2R_3 + R_2 \to R_2$, $-5R_3 + R_1 \to R_1$ $\left[\begin{array}{ccc|c} 1 & 5 & 0 & -14 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$ $-5R_2 + R_1 \to R_1$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$$\vec{x} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$$

2. Given $A=\begin{bmatrix} 2 & -2 \\ -2 & 1 \end{bmatrix}$, find A^{-1}. You can use the formula for inverses of 2x2 matrices to check your work, but use the row-reducing procedure to calculate the inverse. (4 points)

$$A^{-1}=\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1}=\frac{1}{2-4}\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}=-\frac{1}{2}\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}=\begin{bmatrix} -\frac{1}{2} & -1 \\ -1 & -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 2 & -2 & 1 & 0 \\ -2 & 1 & 0 & 1 \end{array}\right] \; R_1+R_2\to R_2 \; \left[\begin{array}{cc|cc} 2 & -2 & 1 & 0 \\ 0 & -1 & 1 & 1 \end{array}\right] \; \begin{matrix} -R_2\to R_2 \\ \frac{1}{2}R_1\to R_1 \end{matrix} \; \left[\begin{array}{cc|cc} 1 & -1 & \frac{1}{2} & 0 \\ 0 & 1 & -1 & -1 \end{array}\right] \; R_2+R_1\to R_1$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -\frac{1}{2} & -1 \\ 0 & 1 & -1 & -1 \end{array}\right]$$

3. Given $A=\begin{bmatrix} 1 & 4 \\ 2 & 1 \\ 0 & -3 \end{bmatrix}, B=\begin{bmatrix} 0 & 5 & -1 \\ 4 & -2 & 0 \end{bmatrix}, C=\begin{bmatrix} 1 & 2 \\ 9 & 7 \\ 0 & 0 \end{bmatrix}$, compute the following, if possible. If the combination is not possible, briefly explain why. (4 points each)

a) AB

$$\begin{bmatrix} 1(0)+4(4) & 1(5)+4(-2) & 1(-1)+4(0) \\ 2(0)+4(1) & 2(5)+1(-2) & 2(-1)+1(0) \\ 0(0)-3(4) & 0(5)-3(-2) & 0(-1)-3(0) \end{bmatrix}$$

$$=\begin{bmatrix} 16 & -3 & -1 \\ 4 & 8 & -2 \\ -12 & 6 & 0 \end{bmatrix}$$

b) AC

not defined

$(3\times 2)\cdot(3\times 2)$

does not match

c) B^T

$$\begin{bmatrix} 0 & 4 \\ 5 & -2 \\ -1 & 0 \end{bmatrix}$$

d) 2C + A

$$\begin{bmatrix} 2 & 4 \\ 18 & 14 \\ 0 & 0 \end{bmatrix}+\begin{bmatrix} 1 & 4 \\ 2 & 1 \\ 0 & -3 \end{bmatrix}=\begin{bmatrix} 3 & 8 \\ 20 & 15 \\ 0 & -3 \end{bmatrix}$$

4. Determine if $\vec{x} = \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}$ is a solution to the system $\begin{cases} x_1 \quad\quad\quad -3x_3 = 8 \\ 2x_1 + 2x_2 + 9x_3 = 7 \\ \quad\quad x_2 + 5x_3 = -2 \end{cases}$. (5 points)

$$\begin{bmatrix} 1 & 0 & -3 \\ 2 & 2 & 9 \\ 0 & 1 & 5 \end{bmatrix} \begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 5+0+3 \\ 10+6-9 \\ 0+3-5 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ -2 \end{bmatrix}$$

yes, it is a solution to the system

5. Using the vectors $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *and* $\vec{v} = \begin{bmatrix} -4 \\ 1 \end{bmatrix}$, determine the result geometrically of applying the transformation $T(\vec{x}) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ from $R^2 \to R^2$. Sketch the before and after vectors. [Hint: Two operations are going on here. It may help to factor out an appropriate size scalar.] (8 points)

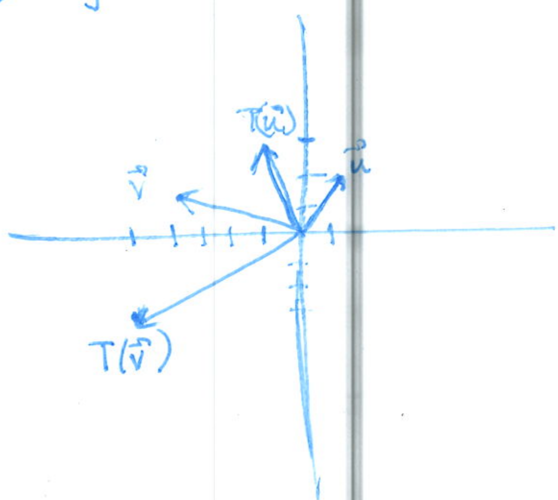
$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1-2 \\ 1+2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 1 \end{bmatrix} = \begin{bmatrix} -4-1 \\ -4+1 \end{bmatrix} = \begin{bmatrix} -5 \\ -3 \end{bmatrix}$$

$$T = \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} =$$

$$\sqrt{2} \begin{bmatrix} \cos \pi/4 & -\sin \pi/4 \\ \sin \pi/4 & \cos \pi/4 \end{bmatrix}$$

scales by $\sqrt{2}$ both x and y
rotates by 45° or $\pi/4$



MTH 266, Exam #1, Part II, Fall 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available. On this portion of the exam, you *may* use a calculator to perform elementary matrix operations. Support your answers with work (reproduce the reduced matrices from your calculator) or other justification for full credit.

6. Determine if each statement is True or False. (1 point each)

a. T F The homogeneous system $\mathbf{Ax} = \mathbf{0}$ is always consistent.

b. T F If *A* is a $m \times n$ matrix that has *m* pivot columns, then the equation $A\mathbf{x} = \mathbf{b}$ is consistent for all **b** in $\Re^m$.

c. T F Any set of vectors containing the zero vector is linearly independent.

dependent

d. T F Matrix multiplication is commutative.

e. T F A 5×7 matrix has 7 columns.

f. T F If a system of equations has a free variable then it has a unique solution.

dependent

g. T F If A is a $n \times n$ matrix, then A is invertible.

h. T F If *A* is a $m \times n$ matrix, then $AI_n = A$ and $I_m A = A$.

i. T F $\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix}$ form a linearly dependent set.

multiples of each other

j. T F f: $R \rightarrow R$defined by $f(x) = 2x + 3$ is a linear transformation.

this is affine

7. Find the general solution to the system $\begin{cases} x_1 - 2x_2 + 4x_3 + 5x_4 = 2 \\ \quad\ \ x_2 - 3x_3 - x_4 = 4 \\ \quad 2x_2 - 6x_3 - 8x_4 = 6 \end{cases}$. State whether the solution of the system is consistent or inconsistent. If the system is consistent, state whether it is independent or dependent. Write an independent solution in vector form; write a dependent solution in parametric form. Circle the pivots of the reduced matrix. (8 points)

$$\text{rref} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -2 & 0 & 9 \\ 0 & 1 & -3 & 0 & 13/3 \\ 0 & 0 & 0 & 1 & 1/3 \end{array}\right]$$

consistent
dependent

$$\begin{aligned} x_1 - 2x_3 &= 9 \\ x_2 - 3x_3 &= 13/3 \\ x_3 &= x_3 \\ x_4 &= 1/3 \end{aligned} \Rightarrow \begin{aligned} x_1 &= 2x_3 + 9 \\ x_2 &= 3x_3 + 13/3 \\ x_3 &= x_3 \\ x_4 &= \quad\ \ + 1/3 \end{aligned} \Rightarrow \vec{x} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix} x_3 + \begin{bmatrix} 9 \\ 13/3 \\ 0 \\ 1/3 \end{bmatrix}$$

8. Determine if $\mathbf{b} = \begin{bmatrix} -7 \\ 9 \\ 14 \end{bmatrix}$ is in the span of the columns of $A = \begin{bmatrix} 1 & -4 & -14 \\ 0 & 3 & 12 \\ -2 & 8 & 28 \end{bmatrix}$.

If it is, write **b** as a linear combination of the columns of *A*; if not, explain why it is not. (8 points)

$$\left[\begin{array}{ccc|c} 1 & 4 & -14 & -7 \\ 0 & 3 & 12 & 9 \\ -2 & 8 & 28 & 14 \end{array}\right] \Rightarrow \text{rref} \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 5 \\ 0 & 1 & 4 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

yes, it is in the span

$$\vec{b} = 5\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + 3\begin{bmatrix} -4 \\ 3 \\ 8 \end{bmatrix} + 0\begin{bmatrix} -14 \\ 12 \\ 28 \end{bmatrix}$$

9. Let A = $\begin{bmatrix} 1 & 4 & -3 & 0 \\ -2 & -7 & 5 & 1 \\ -4 & -5 & 7 & 5 \end{bmatrix}$

a. Determine if the columns of A form a linearly independent or dependent set and justify your answer. (4 points)

They do not, they cannot since there are more columns than rows

moreover rref $\Rightarrow \begin{bmatrix} 1 & 0 & 0 & -3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

no pivot in this column

b. Determine if the columns of A span R^3. Justify your answer. (4 points)

they do span $\mathbb{R}^3$: matrix above in reduced form has a pivot in every row

10. Given $T: R^3 \rightarrow R^3$ such that $T(\vec{x}) = A\vec{x}$ and A is given by $A = \begin{bmatrix} 5 & 1 & -1 \\ 1 & -1 & 3 \\ 0 & 7 & 4 \end{bmatrix}$, answer the following.

rref $\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

a. Is T onto R^3? Justify your answer. (4 points)

yes, each row has a pivot, Therefore onto

b. Is T one-to-one? Justify your answer. (4 points)

yes, each column has a pivot, Therefore one-to-one

11. Consider the transformation $T: P_n \to P_{n+1}$ such that $T(f) = \int_0^t f(x)dx$. If $f(x)$ is any polynomial in P_n, use the definition of a linear transformation to show that T is linear. (10 points)

is it the case that $T(f+g) = T(f) + T(g)$? yes

Since $T(f+g) = \int_0^t (f(x)+g(x))dx = \int_0^t f(x)dx + \int_0^t g(x)dx = T(f)+T(g)$

by properties of integrals

is it the case that $T(kf) = kT(f)$? yes

Since $T(kf) = \int_0^t kf(x)dx = k\int_0^t f(x)\,dx = kT(f)$

by properties of integrals

and $T(0) = \int_0^t 0\,dx = 0$.

12. **Use an inverse matrix** to solve $\begin{cases} x_1 \quad\quad - 2x_3 = 1 \\ -3x_1 + x_2 + 4x_3 = -5 \\ 2x_1 - 3x_2 + 4x_3 = 8 \end{cases}$ (8 points)

$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 1 \\ -5 \\ 8 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}$

$A^{-1}\vec{b} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ -5 \\ 8 \end{bmatrix} = \begin{bmatrix} 8-15+8 \\ 10-20+8 \\ 7/2-15/2+8/2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \vec{X}$

13. The invertible matrix theorem states that several statements are equivalent to matrix A being invertible. Name 5 of these equivalent statements. (10 points)

answers will vary (see list in book) but could include:
if A is $n \times n$, then:

$A\vec{x} = 0$ has only the trivial solution

A has a pivot in every column

A has a pivot in every row

A reduces to the identity (row-equivalent)

the columns of A are linearly independent

etc.

14. Answer the following questions as fully as possible, and justify your answer. (4 points each)

a. If a linear transformation $T: \mathrm{R}^{\mathrm{n}} \rightarrow \mathrm{R}^{\mathrm{m}}$ maps R^n onto R^m, can you give a relationship between m and n?

$m \times n$

$m \leq n$

b. If T is *one-to-one* what can you say about m and n?

$n \leq m$

c. Explain why the columns of A^2 (defined by matrix multiplication as AA) span R^n whenever the columns of an $n \times n$ matrix A are linearly independent.

answers may vary

if A is $n \times n$, then $AA = A^2$ is defined

if A has linearly independent columns then A is invertible and transformation defined by A is one-to-one and onto. Thus A^2 is also one-to-one and onto, and A^2 is also invertible

if A^2 is invertible then its columns are linearly independent also

d. How many pivot columns must a 6×4 matrix have if its columns span R^4.

4

e. If A is a 2×5 matrix with two pivot positions, does the equation $A\vec{x} = \vec{0}$ have a solution? If so, is it trivial or non-trivial?

2 pivots $\Rightarrow$ 3 free variables

$A\vec{x} = \vec{0}$ has a solution, but it is non-trivial