MTH 267, Exam #2, Part I, Fall 2018 Name KEY

Instructions: Show all work. Give exact answers unless specifically asked to round. All complex numbers should be stated in standard form, and all complex fractions should be simplified. If you do not show work, problems will be graded as "all or nothing" for the answer only; partial credit will not be possible and any credit awarded for the work will not be available.

1. A force of 30 N stretches a spring 0.6 meters. A mass of 40 kg is attached to the end of the spring and is initially released from equilibrium position with an upward velocity of 0.2 m/s. Write the second-order equation that models the system. (7 points)

$F = kx$

$30 = k(.6)$

$k = 50$

$m = 40$

$$40y'' + 50y = 0$$

$y(0) = 0$

$y'(0) = 0.2$

a. A dashpot device is added to the system to provide damping numerically equivalent to twice the velocity. Write the new second-order system. (5 points)

$$40y'' + 2y' + 50y = 0 \Rightarrow y'' + \tfrac{1}{20}y' + \tfrac{5}{4}y = 0$$

b. Convert the second-order equation in (a) to a first order system. Use technology to graph the phase plane. Use this information to determine if the system is underdamped or overdamped. (6 points)

$y = x_1$

$y' = x_1' = x_2$

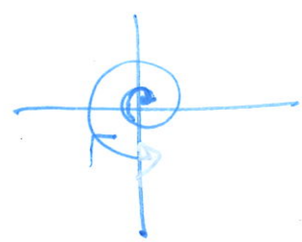
$y'' = x_2'$

$x_2' = -\frac{1}{20}x_2 - \frac{5}{4}x_1$

$x_1' = x_2$

$x_2' = -\frac{5}{4}x_1 - \frac{1}{20}x_2$

underdamped



c. Solve the system. (12 points)

$$\begin{pmatrix} 0 & 1 \\ -5/4 & -\frac{1}{20} \end{pmatrix} \rightarrow \begin{pmatrix} -\lambda & 1 \\ -5/4 & -\frac{1}{20}-\lambda \end{pmatrix}$$

$$-\lambda\left(-\tfrac{1}{20}-\lambda\right) + \tfrac{5}{4} = \lambda^2 + \tfrac{1}{20}\lambda + \tfrac{5}{4} = 0$$

$$\lambda = \frac{-1}{40} \pm \frac{\sqrt{1999}}{40}i$$

$$\begin{pmatrix} \frac{1}{40} - \frac{\sqrt{1999}}{40}i & 1 \\ -5/4 & -\frac{1}{40} - \frac{\sqrt{1999}}{40}i \end{pmatrix}$$

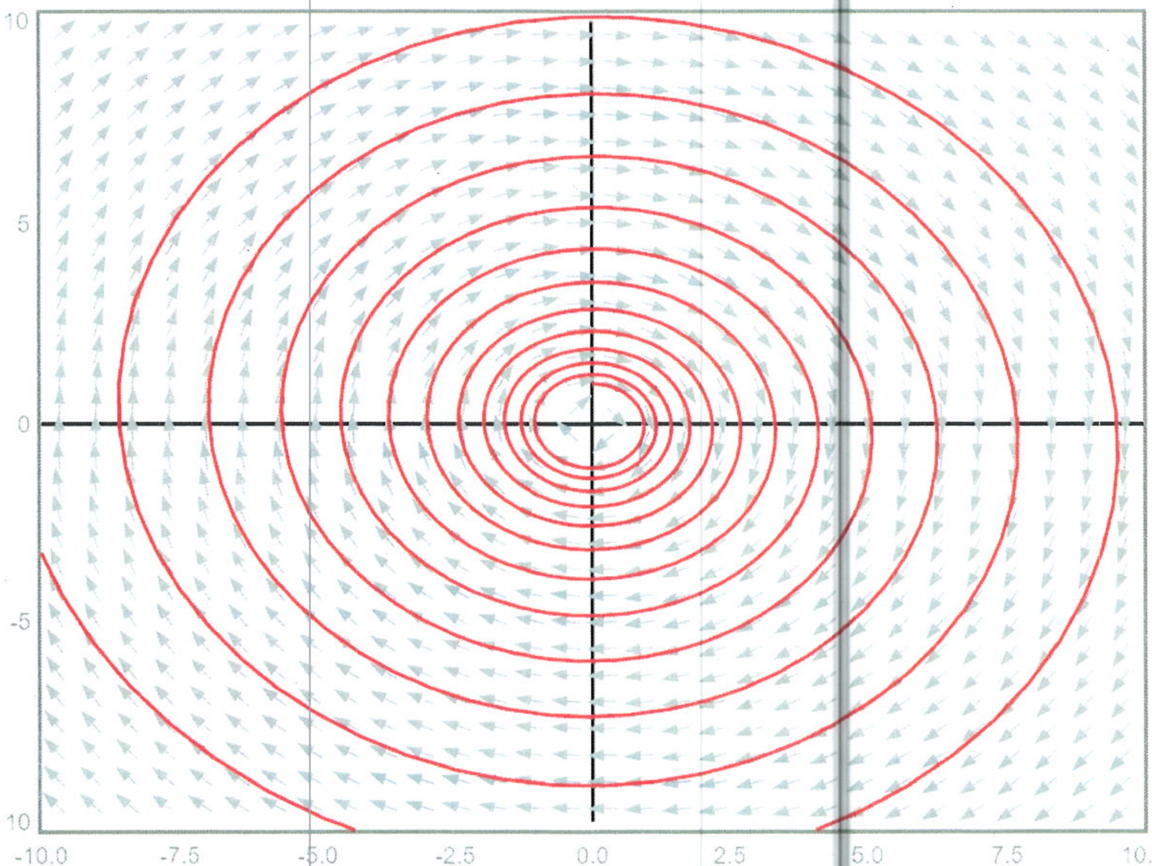
$$-\tfrac{4}{5}\cdot -\tfrac{5}{4}x_1 = \left(\tfrac{1}{40} + \tfrac{\sqrt{1999}}{40}i\right)x_2 \cdot -\tfrac{4}{5}$$

$$x_1 = -\left(\tfrac{1}{50} + \tfrac{\sqrt{1999}}{50}i\right)x_2$$

$$\begin{pmatrix} 1+\sqrt{1999}\,i \\ -50 \end{pmatrix}$$

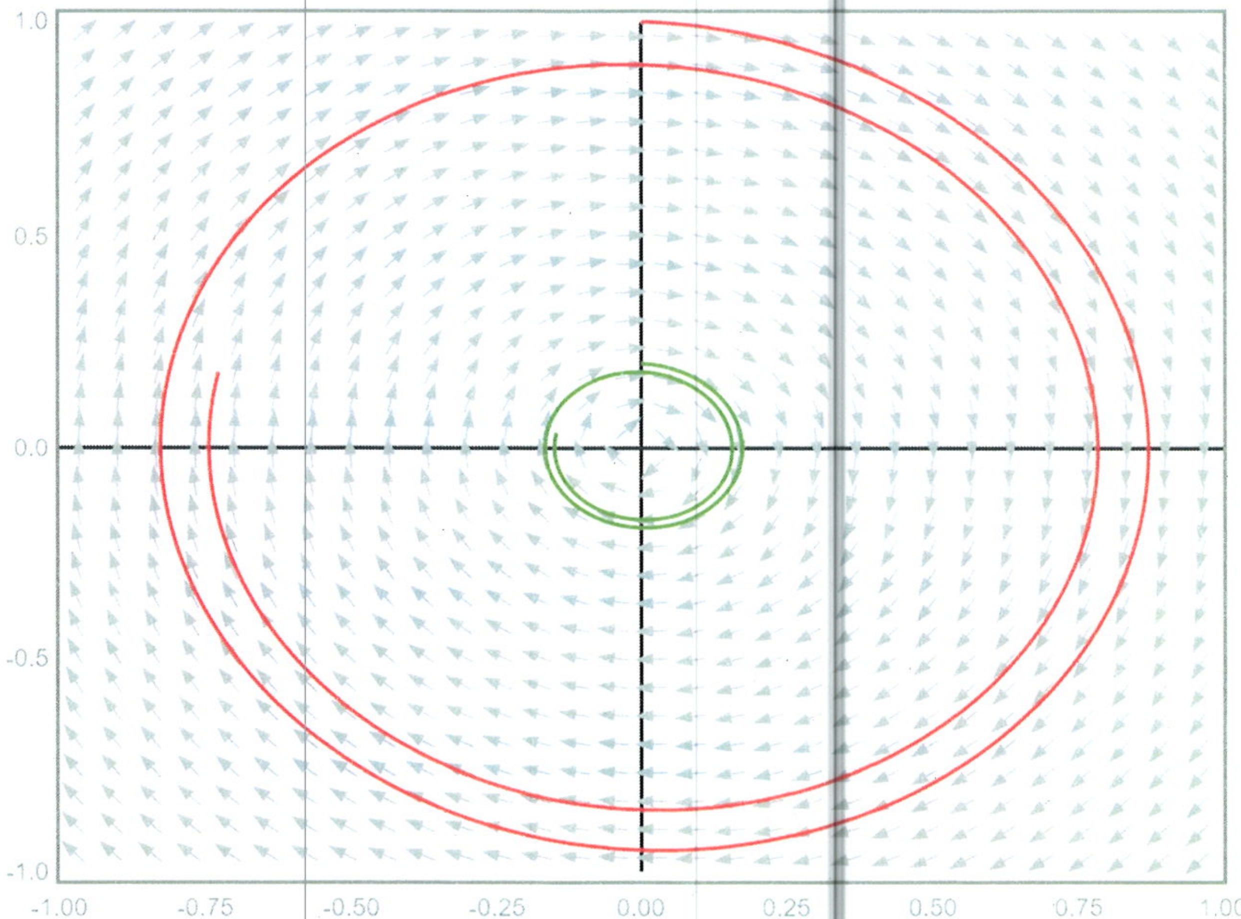
$$\begin{pmatrix} 1+\sqrt{1999}\,i \\ -50 \end{pmatrix} e^{-t/40}\left(\cos\left(\tfrac{\sqrt{1999}}{40}t\right) + i\sin\left(\tfrac{\sqrt{1999}}{40}t\right)\right) = e^{-t/40}\begin{pmatrix} \cos\left(\frac{\sqrt{1999}}{40}t\right) + i\sin\left(\frac{\sqrt{1999}}{40}t\right) + \sqrt{1999}\,i\cos\left(\frac{\sqrt{1999}}{40}t\right) - \sqrt{1999}\sin\left(\frac{\sqrt{1999}}{40}t\right) \\ -50\cos\left(\frac{\sqrt{1999}}{40}t\right) - i50\sin\left(\frac{\sqrt{1999}}{40}t\right) \end{pmatrix}$$

$\downarrow$



$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}' = c_1 e^{-t/40} \begin{pmatrix} \cos\left(\frac{\sqrt{1999}}{40}t\right) - \sqrt{1999}\sin\left(\frac{\sqrt{1999}}{40}t\right) \\ -50\cos\left(\frac{\sqrt{1999}}{40}t\right) \end{pmatrix} +$$

$$c_2 e^{-t/40} \begin{pmatrix} \sin\left(\frac{\sqrt{1999}}{40}t\right) + \sqrt{1999}\cos\left(\frac{\sqrt{1999}}{40}t\right) \\ -50\sin\left(\frac{\sqrt{1999}}{40}t\right) \end{pmatrix}$$



inner circle is for our initial
conditions

d. Sketch the graph of the solution for the given initial conditions. (6 points)

See attached

2. Consider the competition model $\begin{cases} \frac{dx}{dt} = -0.85x + 0.5x^2 + xy \\ \frac{dy}{dt} = 0.5y - 0.1y^2 - xy \end{cases}$. Sketch the nullclines by hand for the system and use them to identify any equilibria. Using the information obtained from the nullclines, and a technology-generated full phase plane, can you characterize the equilibria as stable, unstable or a saddle point? Be sure to attach all your graphs. (20 points)

$0 = -0.85x + 0.5x^2 + xy = x(-0.85 + 0.5x + y) = 0$

$x = 0$ $\quad$ $y = -0.5x + 0.85$

$0 = 0.5y - 0.1y^2 - xy = y(0.5 - 0.1y - x) = 0$

$y = 0$ $\quad$ $x - 0.5 = -0.1y \; (-10)$

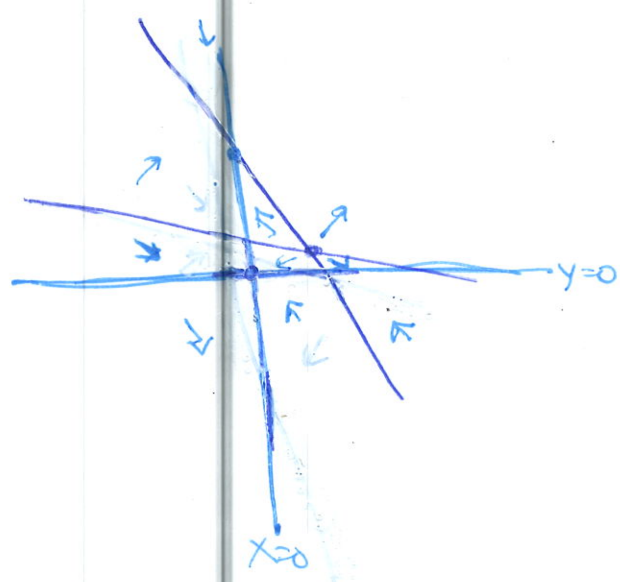
$-10x + 5 = y$

$-0.5x + 0.85 = -10x + 5$

$9.5x = 4.15$

$x = \frac{83}{190} \approx .4368$

$y = \frac{203}{190} \approx 1.068$



equilibria

$(0,0)$ Saddle

$\left(\frac{83}{190}, \frac{203}{190}\right)$ unstable

$(0,5)$ saddle

$\left(\frac{17}{10}, 0\right)$ Saddle

3. Consider the second order differential equations shown below. Convert the second-order equations to systems, and then graph their phase planes using technology. Use the information obtained from the graph to determine the damping of the system (if it can model a spring system), or if the equation cannot be a spring model. Explain your reasoning in each case. (12 points)

a. $y'' + 6y' + 9y = 0$

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -9 & -6 \end{pmatrix} \vec{x}$$

overdamped
converges to origin

b. $y'' - 11y' + 24y = 0$

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -24 & 11 \end{pmatrix} \vec{x}$$

not a possible spring
$-11y'$ adds energy
diverges from origin

c. $y'' + 4y = 0$

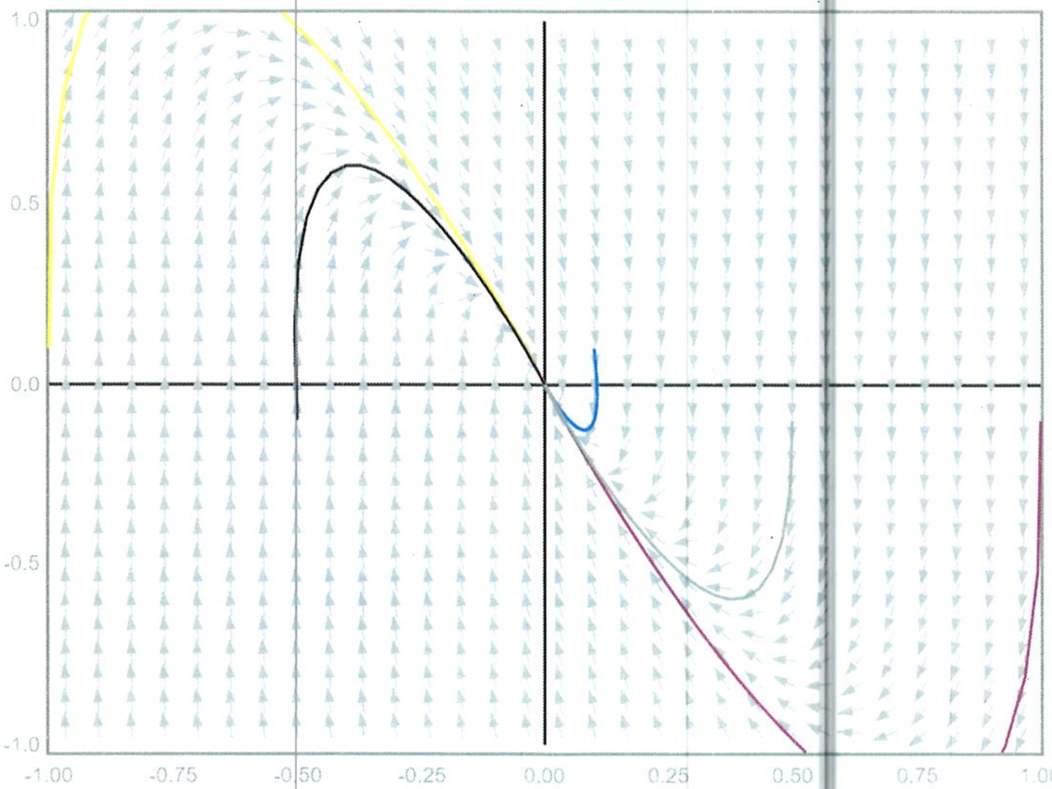
$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} \vec{x}$$

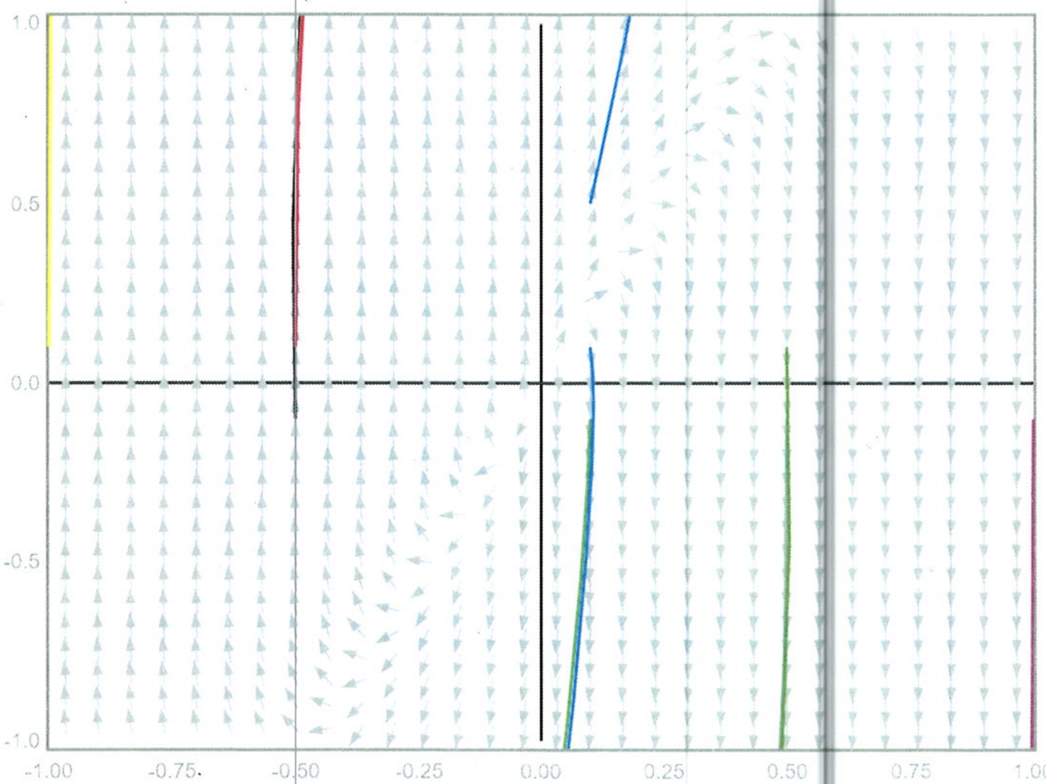
undamped
stable orbit

d. $y'' + 2y' + 11y = 0$

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -11 & -2 \end{pmatrix} \vec{x}$$

underdamped
converges to origin
(spirals)


1.0
0.5
0.0
-0.5
-1.0
-1.00
-0.75
-0.50
-0.25
0.00
0.25
0.50
0.75
1.00
3a


1.0
0.5
0.0
-0.5
-1.0
-1.00
-0.75
-0.50
-0.25
0.00
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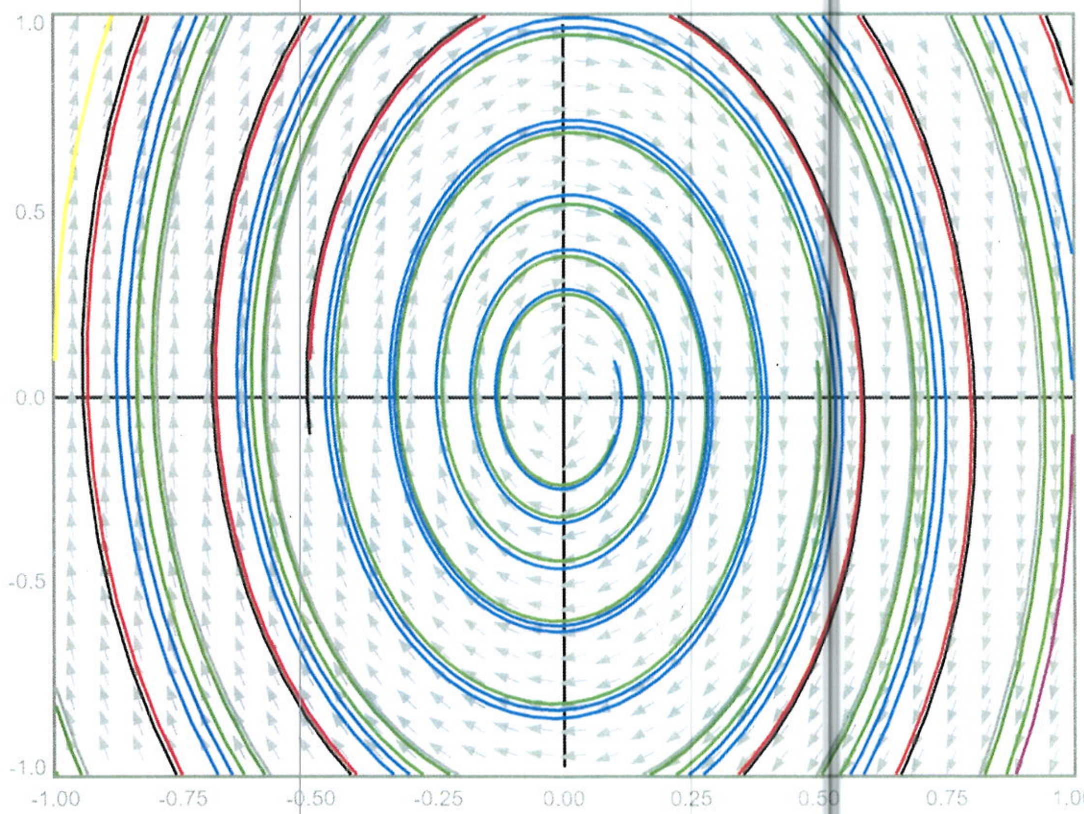
0.50
0.75
1.00
3b


1.0
0.5
0.0
-0.5
-1.0
-1.00
-0.75
-0.50
-0.25
0.00
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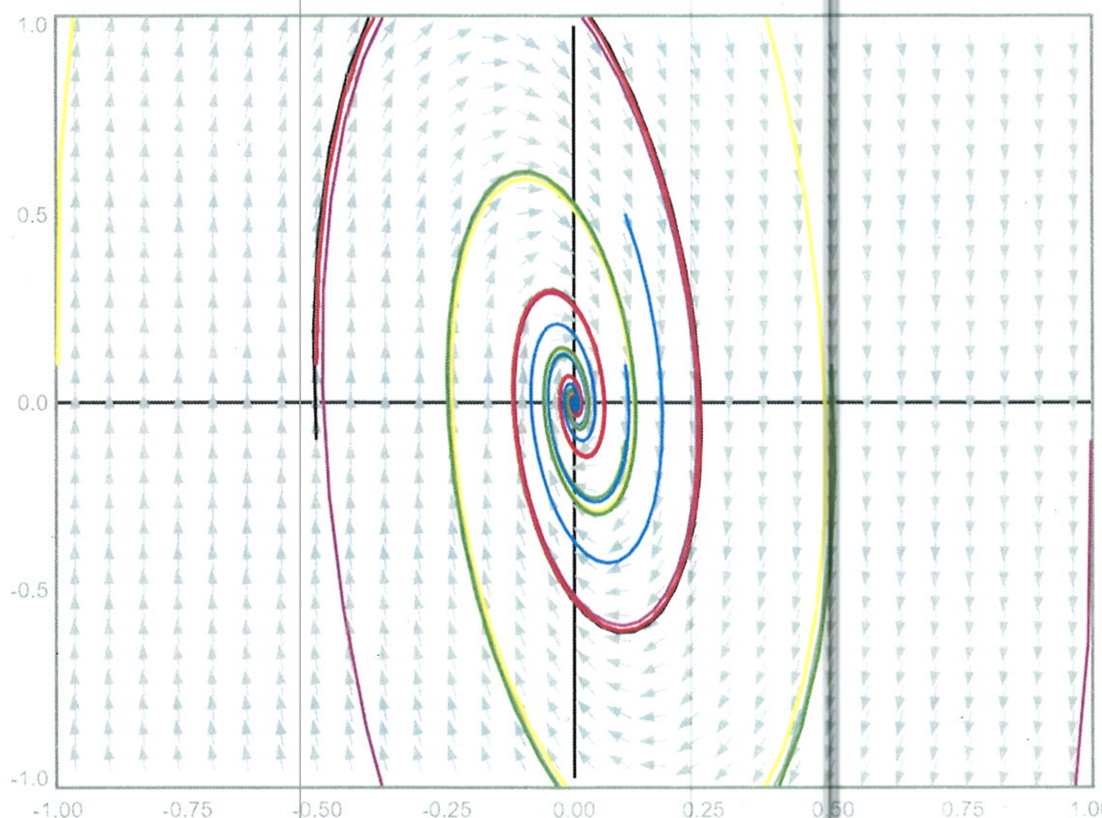
0.50
0.75
1.00

3c


1.0
0.5
0.0
-0.5
-1.0
-1.00
-0.75
-0.50
-0.25
0.00
0.25
0.50
0.75
1.00

3d

MTH 267, Exam #2, Part II, Fall 2018 Name KEY

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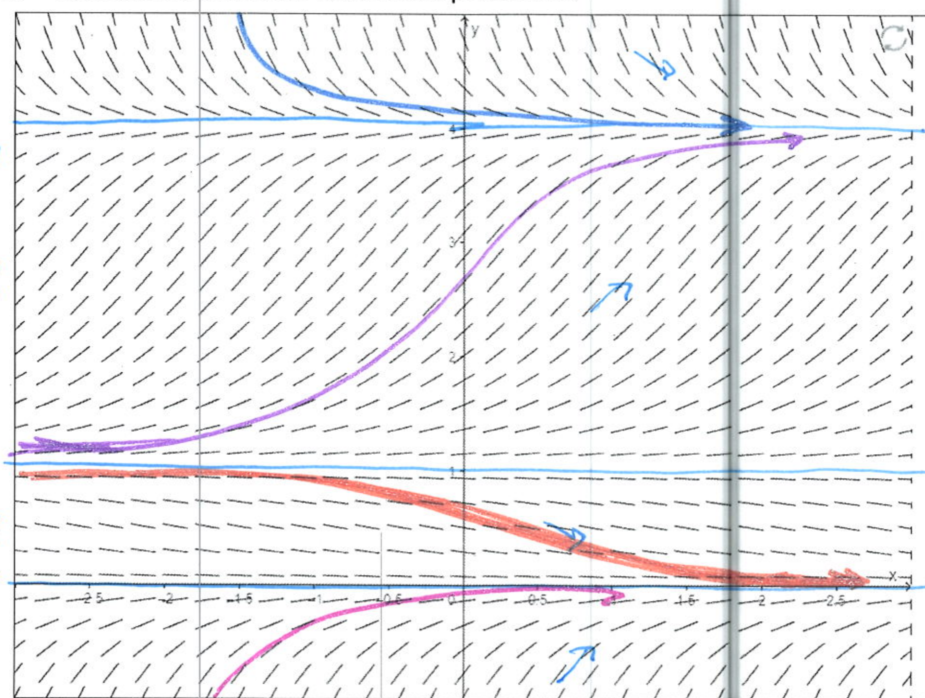
4. Direction fields for population models are shown below. (12 points)
   a. Find a differential equation that models the population (up to a constant multiple).
   b. Plot trajectories of initial conditions that models each type of trajectory for the model.
   c. Where is the model logistic?
   d. Describe the long-term behavior of each trajectory.
   e. Describe each equilibrium as a carrying capacity or a threshold.
   f. Convert the direction field into a phase line.

goes to 4

logistic

logistic

goes to 0



y=4 Carrying capacity

goes to 4

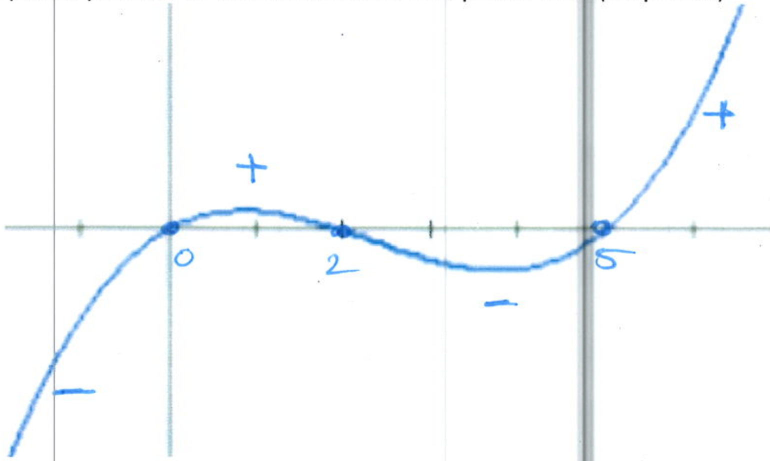
y=1 Threshold

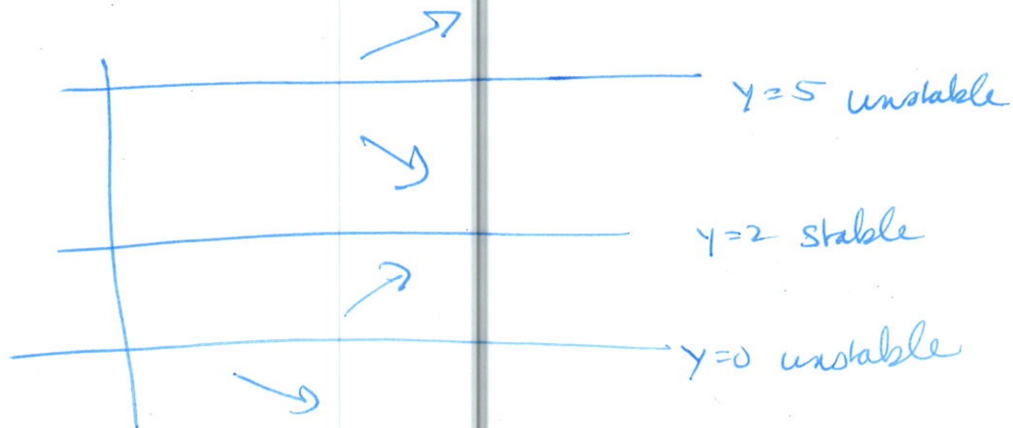
goes to 0

y=0

$$\frac{dy}{dt} = -y(y-1)(y-4)$$

5. The phase portrait for the differential equation $y' = 0.1y\left(1-\frac{y}{5}\right)\left(\frac{y}{2}-1\right)$ is shown below. Convert the phase portrait to a direction field and phase line. (12 points)





6. Solve the systems of equations for the general solution below using eigenvalues. Be sure that your solutions are expressed only with real-valued functions. (14 points each)

a. $\vec{x}'(t) = \begin{pmatrix} 3 & 2 \\ 3 & 4 \end{pmatrix}\vec{x}$

$(3-\lambda)(4-\lambda)-6=0$

$\lambda^2-7\lambda+12-6=0$

$\lambda^2-7\lambda+6=0$

$(\lambda-6)(\lambda-1)=0$

$\lambda=6,\ \lambda=1$

$\lambda=6$ $\quad \begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix}$ $\quad 3x_1=2x_2$, $x_1=\frac{2}{3}x_2$ $\quad \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\lambda=1$ $\quad \begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix}$ $\quad x_1=-x_2$ $\quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\vec{x} = c_1\begin{pmatrix} 2 \\ 3 \end{pmatrix}e^{6t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{t}$

b. $\vec{x}'(t) = \begin{pmatrix} -8 & 9 \\ -5 & 4 \end{pmatrix} \vec{x}$

$(-8-\lambda)(4-\lambda)+45=0$

$\lambda^2+4\lambda-32+45=0$

$\lambda^2+4\lambda+13=0$

$\lambda = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm 6i}{2} \Rightarrow$

$= -2 \pm 3i$

$\lambda = -2+3i$

$\begin{pmatrix} -8+2-3i & 9 \\ -5 & 4+2-3i \end{pmatrix} = \begin{pmatrix} -6-3i & 9 \\ -5 & 6-3i \end{pmatrix}$

$5x_1 = (6-3i)x_2$

$x_1 = \frac{6-3i}{5} x_2 \quad \begin{pmatrix} 6-3i \\ 5 \end{pmatrix}$

$\vec{x} \Rightarrow \begin{pmatrix} 6-3i \\ 5 \end{pmatrix} e^{-2t}(\cos 3t + i \sin 3t) =$

$= e^{-2t} \begin{pmatrix} 6\cos 3t + 6i\sin 3t - 3i\cos 3t + 3\sin 3t \\ 5\cos 3t + 5i\sin 3t \end{pmatrix}$

$\vec{x} = c_1 e^{-2t} \begin{pmatrix} 6\cos 3t + 3\sin 3t \\ 5\cos 3t \end{pmatrix} +$

$c_2 e^{-2t} \begin{pmatrix} 6\sin 3t - 3\cos 3t \\ 5\sin 3t \end{pmatrix}$

7. A cup of water originally at a temperature of 120° is taken outside and left to cool. The outside temperature is 50°. After one minute, the temperature of the water has dropped to 100°. Write a differential equation to model the system, and then solve the system. How much time is required for the water to drop to 55°? (12 points)

$T(0)=120 \quad T(1)=100°$

$u=50$

$\frac{dT}{dt} = k(T-50)$

$\int \frac{dT}{T-50} = \int k\,dt$

$\ln|T-50| = kt + C$

$T-50 = e^{kt} T_0$

$T(t) = 50 + T_0 e^{kt}$

$120 = 50 + T_0 (e^0)^1$

$T_0 = 70$

$T(t) = 50 + 70e^{kt}$

$100 = 50 + 70e^k$

$50 = 70e^k$

$\frac{5}{7} = e^k \Rightarrow \ln\frac{5}{7} = k$

$T(t) = 50 + 70e^{-.336472t}$

$55 = 50 + 70e^{-.336472t}$

$\frac{5}{70} = e^{-.336472t}$

$t = \frac{\ln\left(\frac{5}{70}\right)}{-.336472} \approx 7.84$ minutes