MTH 266, IOLA Homework, Unit 2 Name ____

Instructions: Answer the following questions and attach your answers to this page for submission.

1. Let $A = \begin{bmatrix} 3 & 0 \\ 0 & -2 \end{bmatrix}$ and define $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\boldsymbol{x}) = A\boldsymbol{x}$. Find the images under T of $\boldsymbol{u} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$, $\boldsymbol{v} = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$, and the vector $(\boldsymbol{u} + \boldsymbol{v})$. Also sketch the three vectors before and after the transformation. Write 1-2 sentences that explain how you found your answers/knew they were right and why the sketch makes sense.

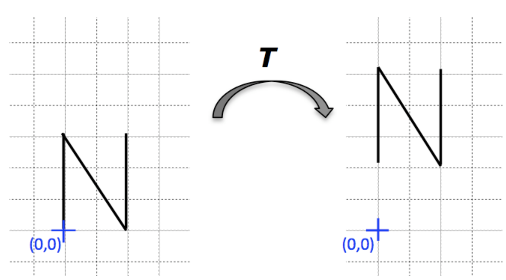
2. Let $A = \begin{bmatrix} 1 & -5 & -7 \\ -3 & 7 & 5 \end{bmatrix}$ and define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\boldsymbol{x}) = A\boldsymbol{x}$.
   a. Find the image under T of $\boldsymbol{u} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.
   b. Find a vector $\boldsymbol{x}$ whose image under T is $\boldsymbol{b} = \begin{bmatrix} -12 \\ 12 \end{bmatrix}$. Explain why your work makes sense.

3. Suppose that a transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. In order to do this, you must show that the definition of linear transformation is satisfied with this matrix A. To help you get started, let $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, and c be a real number. You need to algebraically show both that $A(\boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y}$ and $A\,c\boldsymbol{x} = cA(\boldsymbol{x})$.

4. Show that the transformation T defined by $T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + 5 \\ x_2 \end{bmatrix}$ is not a linear transformation.

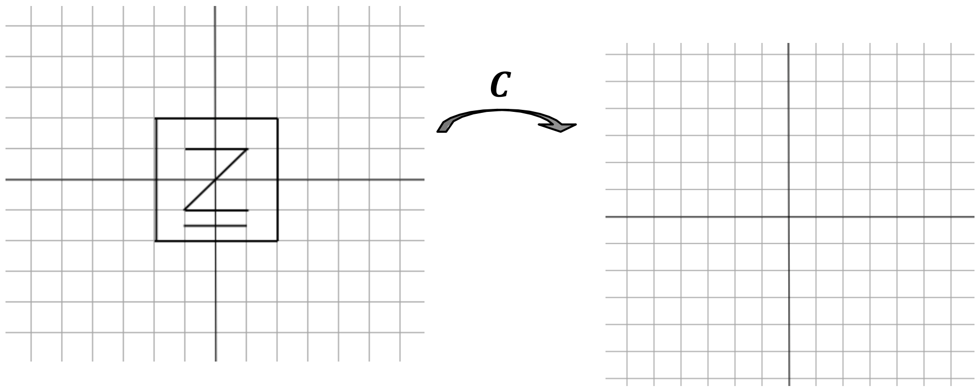
5. True or false: (Assume that the product AB is defined). If the columns of B are linearly dependent, then so are the columns of AB. If TRUE, provide a justification. If FALSE, provide a counterexample.

6. After class, two linear algebra students start talking about linear transformations and the letter "N." One of the students suggested translation (shifting up) as another linear transformation that could be done to the letter "N," like the following:



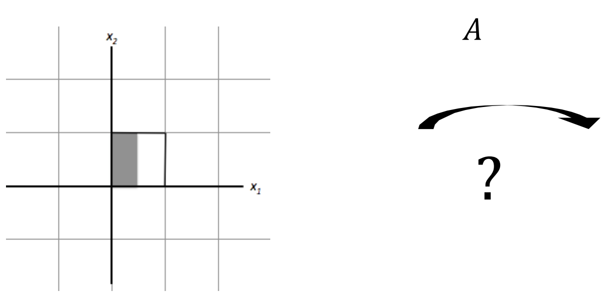
   The other student disagreed, stating that shifting the "N" up like this is NOT an example of a linear transformation. Which student is right? Why?

7. Consider the image given below and the transformation matrix $C = \begin{bmatrix} 2 & 0 \\ 0 & -1.5 \end{bmatrix}$



   a. Sketch what will happen to the image under the transformation.
   b. Describe in words what will happen to the image under the transformation.
   c. Describe how you determined that happened. (What, if any, calculations did you do? Did you make a prediction? How did you know you were right? etc.

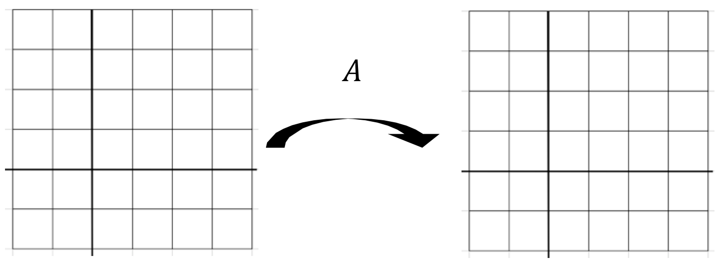
8. Assume that T is a linear transformation and that $\boldsymbol{e_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, and\ \boldsymbol{e_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. For each part, find the standard matrix A for T, and draw the image of the "half-shaded unit square" (shown below) under the given transformation.
   a. $T: \mathbb{R}^2 \to \mathbb{R}^2$ rotates points (about the origin) through $-\frac{\pi}{4}$ radians (clockwise)
   b. $T: \mathbb{R}^2 \to \mathbb{R}^2$ is a vertical shear that maps $\mathbf{e}_1$ into $\mathbf{e}_1 - \mathbf{e}_2$ but leaves the vector $\mathbf{e}_2$ unchanged
   c. $T: \mathbb{R}^2 \to \mathbb{R}^2$ first reflects points across the vertical axis and then rotates points $\frac{\pi}{2}$ radians (counterclockwise)



9. Suppose that $T: \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $T(\boldsymbol{x}) = A\boldsymbol{x}$, where $A = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$.
   a. Find the images of the following vectors under T:

$$\boldsymbol{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \boldsymbol{u} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

b. Sketch the original vectors on one graph and the images of the vectors on a second graph. Do something like the following, where the graph on the left sketches $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{w}, \boldsymbol{u}$, and $\boldsymbol{v}$, and the graph on the graph sketches their images under the transformation.



c. Describe in words what the transformation defined by A does to the vectors. What does the transformation do to the entire space? How did you reach this conclusion?

10. Suppose that $T: \mathbb{R}^n \to \mathbb{R}^m$ is defined by $T(\boldsymbol{x}) = A\boldsymbol{x}$ for each of the matrices listed below. For each given matrix, answer the following questions:

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, E = \begin{bmatrix} 4 & 0 \\ 0 & 0 \\ 0 & 2 \end{bmatrix}, F = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$$

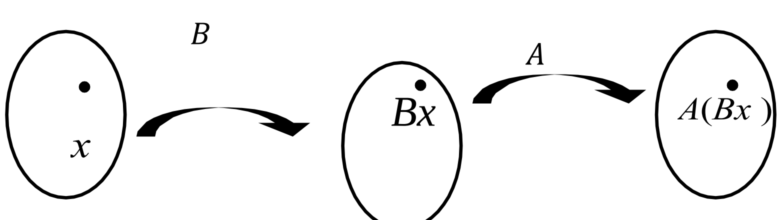
a. Rewrite $T: \mathbb{R}^n \to \mathbb{R}^m$ with correct numbers for m and n filled in for each matrix. What is the domain of T? What is the codomain of T?

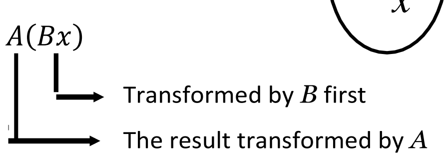
b. Find some way to explain in words and/or graphically what this transformation does in taking vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$. You might find it helpful to try out a few input vectors and see what their image is under the transformation. This might be difficult, but an honest effort will give you credit.

c. For the transformation, can you find two different input vectors that will give the same output vector?
   i. If so, give an example.
   ii. If not, why do you think it isn't possible?

d. For the transformation, can you get any output vector? (Any vector in $\mathbb{R}^m$)
   i. If so, explain why you can get any vector in $\mathbb{R}^m$.
   ii. If not, give an example of an output vector you can't get with the transformation and explain why.

11. In class, we discussed matrix multiplication, when wanting to consider what happens to vectors or the whole space under the transformations, as composition of functions. We thought about how, if you have $AB\boldsymbol{x}$, where A and B are matrices and $\boldsymbol{x}$ is a vector, $AB\boldsymbol{x}$ could be thought of as B transforming $\boldsymbol{x}$ first, and then A transforming the result of $B\boldsymbol{x}$.





Now consider the composition of real-valued functions, familiar from high school. Use the given functions f, g, and h below and carry out the requested computations.

$$f(x) = 2x + 4, \qquad g(x) = x^2 - 3x, \qquad h(x) = \frac{\sqrt{x}}{x - 2}$$

a. $f(g(x))$ and $g(f(x))$
b. $f(g(2))$ and $g(f(2))$
c. $h(f(x))$
d. $h(f(0))$ and $h(f(-1))$
e. $f\left(g(h(x))\right)$
f. Reflect on how composing these functions and evaluating at a value (such as at 2 in part b) is similar to what we saw with matrix multiplication through the Pat & Jamie Task on Wednesday. Write at least two thoughtful sentences.

12. Let the matrices F and G be defined as below. Answer the following questions accordingly.

$$F = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 0 \\ 0 & 3 & 4 \end{bmatrix}, \qquad G = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 3 & -2 \\ 5 & 0 & 1 \end{bmatrix}$$

a. Let $\boldsymbol{x} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, and let $G\boldsymbol{x} = \boldsymbol{y}$. Compute $G\boldsymbol{x}$ and compute $F\boldsymbol{y}$.
b. Let $x = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, and let $F\boldsymbol{x} = \boldsymbol{u}$. Compute $F\boldsymbol{x}$ and compute $G\boldsymbol{u}$.
c. Explain how parts a) and b) illustrate matrix multiplication as composition of functions.
d. Explain how parts a) and b) illustrate that matrices F and G are not commutative.

13. Let A be a 5×4 matrix and B be a 4×3 matrix.
a. What is the domain of the transformation defined by B? What is the codomain of the transformation defined by B?
b. What is the domain of the transformation defined by A? What is the codomain of the transformation defined by A?
c. What is the domain of the transformation defined by AB? What is the codomain of the transformation defined by AB?
d. Explain why the product BA is not defined. Ground your explanation in the concepts of composition of functions, domain, and codomain.