Math 266, Exam #1, Fall 2021 Name KEY

Part I:

Instructions: Show all work. You may **not** use a calculator on this portion of the exam. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals). Reduce as much as possible. Be sure to complete all parts of each question. Provide explanations where requested. You must show all hand written work on this part of the exam. Answers with no work will receive only 1 point. When you are finished with this portion of exam, continue with Part II.

1. Write the system of equations $\begin{cases} 2x_1 + 5x_2 = 7 \\ x_1 - 2x_2 = 8 \end{cases}$ as a) a vector equation, b) a matrix equation, c) an augmented matrix. (16 points)

a) $x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 5 \\ -2 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

b) $\begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$

c) $\begin{bmatrix} 2 & 5 & 7 \\ 1 & -2 & 8 \end{bmatrix}$

2. Row reduce the system to obtain the solution $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. (12 points)

$R_1 \leftrightarrow R_2$

$\begin{bmatrix} 1 & -2 & 8 \\ 2 & 5 & 7 \end{bmatrix}$

$-2R_1 + R_2 \to R_2$

$\begin{bmatrix} 1 & -2 & 8 \\ 0 & 9 & -9 \end{bmatrix}$

$\frac{1}{9}R_2 \to R_2$

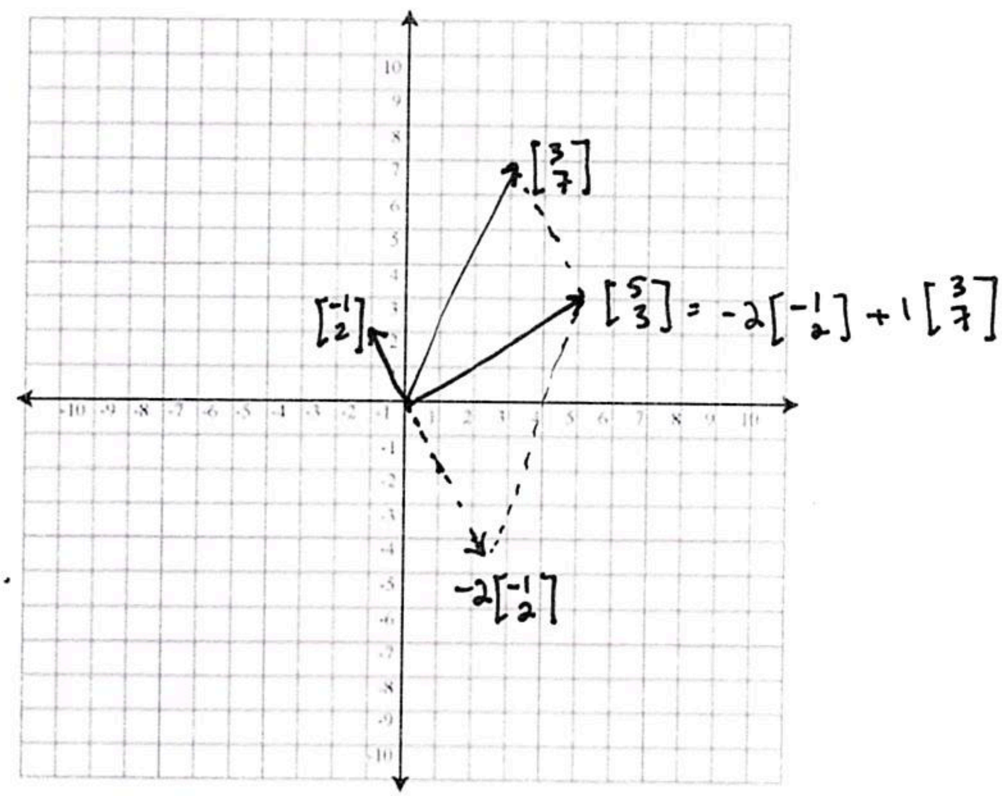
$\begin{bmatrix} 1 & -2 & 8 \\ 0 & 1 & -1 \end{bmatrix}$

$2R_2 + R_1 \to R_1$

$\begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & -1 \end{bmatrix}$

$X = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$

3. The solution to the system $x_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$ is $\vec{x} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. Represent the solution graphically on the graph below. (10 points)



4. Determine if each statement is True or False. (4 points each)

a. T F Two matrices are row equivalent if they have the same dimensions.

b. T F Two fundamental questions about linear systems is about existence and uniqueness.

c. T F Both $\begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 1 & 0 \end{bmatrix}$ are matrices in echelon form.

d. T F The reduced echelon form of a matrix is always unique.

e. T F If two points corresponding to two vectors line on the same line, then the vectors they represent are linearly dependent.

f. T F The $span\{\vec{u}, \vec{v}\}$ is just the lines passing through the point $\vec{u}$ and the origin, and the line passing through the point $\vec{v}$ and the origin.

g. T F The equation $A\vec{x} = \vec{b}$ is consistent if the augmented matrix representing the system has a pivot in every row.

h. T F The solution to the system $A\vec{x} = \vec{b}$ is of the form $\vec{x} = \vec{p} + t\vec{v}$ where $\vec{v}$ is any solution to the system $A\vec{x} = \vec{0}$.

i. T F A homogeneous systems of equations can be inconsistent.

j. T F The pivot rows of a matrix are always linearly independent.

k. T F A linear transformation defined by a 6x4 matrix can be onto, but it cannot be one-to-one.

l. T F A set of vectors are linearly independent if none of the vectors in the set are multiples of any other vector.

m. T F The kernel of a matrix is a subspace of the codomain of the matrix.

n. T F If A is invertible then A is row equivalent to I_n.

o. T F Two matrices, B which is $m \times n$ and C which is $p \times q$, produce a defined product when $m = q$.

p. T F If A has n pivots, then the system $A\vec{x} = \vec{0}$ does not have a unique solution.

5. Determine if the transformation $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 2x_2 \\ x_1 - 3 \\ 2x_1 - 5x_2 \end{bmatrix}$ is linear or not. If it is, prove it. If it is not, find a counterexample. (10 points)

not linear

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0-0 \\ 0-3 \\ 0-0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

addition and scalar multiplication will also fail

6. Give an example of two non-zero matrices whose product is the zero matrix. (8 points)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

7. Consider the following matrices: $A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ -1 & 1 & 1 \end{bmatrix}$, find each of the following matrices or say that they are undefined. If they are undefined, explain why. (8 points each)

a. $C^T + 3I_3$

$$\begin{bmatrix} 1 & 3 & -1 \\ -1 & 0 & 1 \\ 2 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 3 & -1 \\ -1 & 3 & 1 \\ 2 & -2 & 4 \end{bmatrix}$$

b. AB

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 3 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 1-6 & 1+2 & 0-8 \\ 3+3 & 3-1 & 0+4 \end{bmatrix}$$

2x2 2x3

$$= \begin{bmatrix} -5 & 3 & -8 \\ 6 & 2 & 4 \end{bmatrix}$$

c. $B^T A$

$$\begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 1+9 & -2+3 \\ 1-3 & -2-1 \\ 0+12 & 0+4 \end{bmatrix} = \begin{bmatrix} 10 & 1 \\ -2 & -3 \\ 12 & 4 \end{bmatrix}$$

3x2 2x2

8. Find the inverse of the matrix $A = \begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix}$. Use it to solve the system $\begin{bmatrix} -3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 16 \end{bmatrix}$. (10 points)

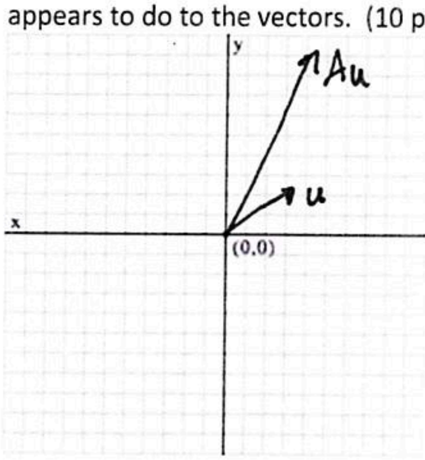
$$A^{-1} = \frac{1}{-3-4} \begin{bmatrix} 1 & -1 \\ -4 & -3 \end{bmatrix} = -\frac{1}{7} \begin{bmatrix} 1 & -1 \\ -4 & -3 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -1 & 1 \\ 4 & 3 \end{bmatrix}$$

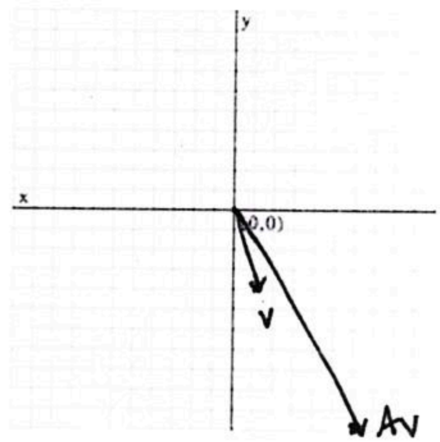
$$A^{-1}b = \frac{1}{7} \begin{bmatrix} -1 & 1 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 16 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} -2+16 \\ 8+48 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} 14 \\ 56 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}$$

Part II:

Instructions: Show all work. You **may** use a calculator on this portion of the exam. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question. Provide explanations where requested.

9. Consider the linear transformation matrix $A = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}$. On the graphs below, graph the vectors $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$, along with $A\vec{u}, A\vec{v}$. Describe in words what the transformation appears to do to the vectors. (10 points)





$$\begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6-2 \\ 3+6 \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ -4 \end{bmatrix} = \begin{bmatrix} 2+4 \\ 1-12 \end{bmatrix} = \begin{bmatrix} 6 \\ -11 \end{bmatrix}$$

lengthens, possible rotation counterclockwise

10. Find the solution to the homogeneous system $\begin{cases} x_1 + 2x_2 - x_3 - 4x_4 + x_5 + 2x_6 = 0 \\ 4x_1 - 3x_2 + 5x_5 - x_6 = 0 \\ 2x_1 - x_3 + 2x_4 + 3x_6 = 0 \end{cases}$. Write the solution in parametric form. (16 points)

$$\begin{bmatrix} 1 & 2 & -1 & -4 & 1 & 2 \\ 4 & -3 & 0 & 0 & 5 & -1 \\ 2 & 0 & -1 & 2 & 0 & 3 \end{bmatrix} \rightarrow \text{rref} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -18/5 & 13/5 & -1 \\ 0 & 1 & 0 & -24/5 & 9/5 & -1 \\ 0 & 0 & 1 & -46/5 & 26/5 & -5 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 18/5\, x_4 - 13/5\, x_5 + x_6 \\ x_2 &= 24/5\, x_4 - 9/5\, x_5 + x_6 \\ x_3 &= 46/5\, x_4 - 26/5\, x_5 + 5x_6 \\ x_4 &= x_4 \\ x_5 &= x_5 \\ x_6 &= x_6 \end{aligned}$$

$$\vec{x} = \begin{bmatrix} 18/5 \\ 24/5 \\ 46/5 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -13/5 \\ -9/5 \\ -26/5 \\ 0 \\ 1 \\ 0 \end{bmatrix} x_5 + \begin{bmatrix} 1 \\ 1 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix} x_6$$

$$\text{or } \vec{x} = \begin{bmatrix} 18 \\ 24 \\ 46 \\ 5 \\ 0 \end{bmatrix} t + \begin{bmatrix} -13 \\ -9 \\ -26 \\ 0 \\ 5 \end{bmatrix} s + \begin{bmatrix} 1 \\ 1 \\ 5 \\ 0 \\ 0 \\ 1 \end{bmatrix} v$$

11. Determine if the following sets of vectors are linearly independent. Explain your reasoning. (25 points)

a. $\left\{ \begin{bmatrix} 1 \\ 5 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \\ 4 \end{bmatrix} \right\}$ yes, independent

2 vectors not multiples of each other

b. $\left\{ \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ yes, independent

2 vectors not multiples of each other

c. $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ yes, independent

reduces to identity / pivot in every column

d. $\{1 - t^2, 3 - 2t, 5t + 7t^2\}$

$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 7 \end{bmatrix}$ yes, independent

reduces to identity / pivot in every column

e. $\{1, 1 - t, (1 - t)^2, (1 - t)^3\}$

$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 3 \\ -1 \end{bmatrix}$ yes, independent

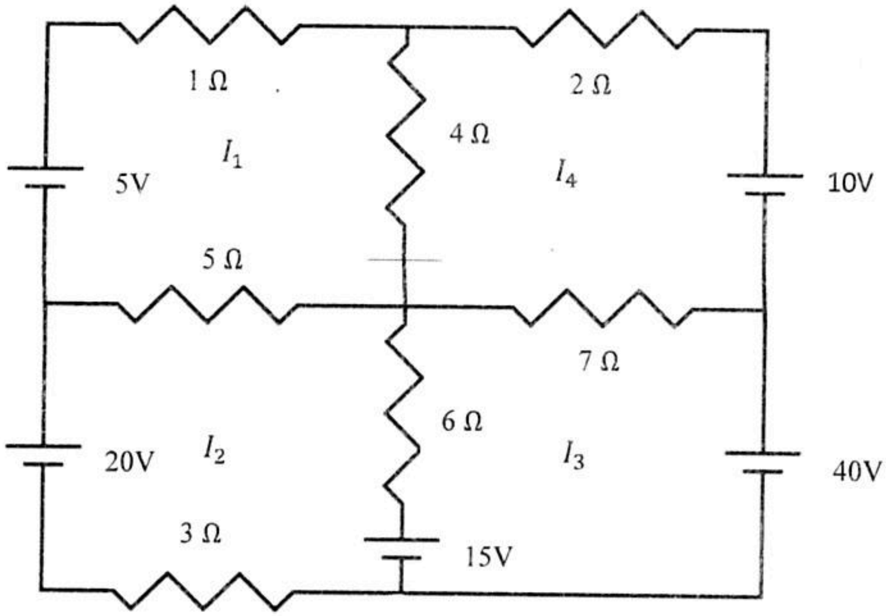
reduces to identity

already in echelon form, so we can see a pivot in every column

$(1-t)^2 = 1 - 2t + t^2$

$(1-t)^3 = 1 - 3t + 3t^2 - t^3$

12. Write a matrix to determine the loop currents and use your calculator to solve the system. Round your answers to two decimal places. (15 points)



$$10 I_1 - 5 I_2 - 4 I_4 = 5$$

$$-5 I_1 + 14 I_2 - 6 I_3 = 20 - 15 = 5$$

$$-6 I_2 + 13 I_3 - 7 I_4 = 15 - 40 = -25$$

$$-4 I_1 \quad - 7 I_3 + 13 I_4 = -10$$

$$\left[\begin{array}{cccc|c} 10 & -5 & 0 & -4 & 5 \\ -5 & 14 & -6 & 0 & 5 \\ 0 & -6 & 13 & -7 & -25 \\ -4 & 0 & -7 & 13 & -10 \end{array}\right] \rightarrow \text{rref} \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -3.45 \\ 0 & 1 & 0 & 0 & -3.64 \\ 0 & 0 & 1 & 0 & -6.46 \\ 0 & 0 & 0 & 1 & -5.31 \end{array}\right]$$

$$I = \begin{bmatrix} -3.45 \\ -3.64 \\ -6.46 \\ -5.31 \end{bmatrix}$$

Solutions can differ by a sign