10/14/2021

Coordinate Systems, Dimensions of a Vector Space

Coordinate Systems

What is a basis? – a basis is a set of vectors with two properties: 1) the set is linearly independent, 2) the vectors span the set.

If you want to span a set of 4 dimensions, such as R^4, $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, then you need a minimum of 4 vectors in the span.

$$span\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\}$$

If we want to span R^4, we need a minimum of 4 vectors, but not a guarantee
For a basis, I also need them to be independent: this ensures that I can get to span all of R^4 with the minimum number of vectors.

The unique representation theorem:
If *B* is a set of basis vectors for V (in V), then for each vector in the space V, there is unique representation for any vector x in V such that $x = c_1 b_1 + c_2 b_2 + \cdots + c_n b_n$.

The coordinate of x in the basis B, $[x]_B$ is the list (vector) of coefficients, the linear combinations of the basis vectors: $\begin{bmatrix} c_1 \\ c_2 \\ \cdots \\ c_n \end{bmatrix}$.

R^2, and the standard basis vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The vector $x = \begin{bmatrix} 4 \\ 7 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

But we can find the coefficients of this point for any set of basis vectors.

New basis for R^2 is $\left\{\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \end{bmatrix}\right\}$.

$$c_1\begin{bmatrix} 2 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$P_B[x]_B = x$$

P_B is the coordinate transformation matrix, and it's just a matrix of the basis vectors (expressed in the standard basis)
$[x]_B$ is the list of coordinates (the vector representation) of x in the new basis.

x is the vector in the standard basis.

$$\frac{1}{13}\begin{bmatrix} 5 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 20+7 \\ -12+14 \end{bmatrix} = \begin{bmatrix} \frac{27}{13} \\ \frac{2}{13} \end{bmatrix} = [x]_B$$

$$\frac{27}{13}\begin{bmatrix} 2 \\ 3 \end{bmatrix} + \frac{2}{13}\begin{bmatrix} -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

What if I have the vector $\begin{bmatrix} 3 \\ -4 \end{bmatrix}_B$. What is the equivalent vector in the standard basis?

$$\begin{bmatrix} 2 & -1 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \end{bmatrix} = \begin{bmatrix} 6+4 \\ 9-20 \end{bmatrix} = \begin{bmatrix} 10 \\ -11 \end{bmatrix}$$

Coordinate transformation are linear transformations. They are both one-to-one and onto. There is always an inverse. The spaces have the same dimension.   They are isomorphic.

The dimension of a vector space is defined as the number of basis vectors used to define the space.

$$span\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 5 \\ 2 \\ -1 \end{bmatrix}\right\}$$

Span is a set of linear combinations of vectors.

Row-reduce: $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, so based on the spanning set theorem, I can drop the dependent vectors to form a basis.

Basis for the subspace is $\left\{\begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right\}$. This subspace is a 2-dimensional subspace of R^4.

Set of polynomials of degree less than or equal to three, P_3. What is the dimension of this space?
This is a 4-dimensional space (isomorphic to R^4). The four (standard) basis vectors for P_3 are $\{1, t, t^2, t^3\}$

For a generic cubic like $at^3 + bt^2 + ct + d$, $\begin{bmatrix} d \\ c \\ b \\ a \end{bmatrix}$.

Column space of a matrix (transformation) is the span of the columns of A. Find a basis for Col A. Row-reduce and pull out the columns of A that have pivots. That tells you the dimension of the column space. These vectors are in R^m. The dimension of the column space is the same as the number of pivots.

Null space of a matrix (transformation) is the span of vectors that solve the homogeneous solution (in parametric form). These vectors are in R^n, they are the same length as the number of columns of the matrix. In the normal way we solve for the null space, each vector from a different free variable is independent of other vectors from other free variables. Those vectors form a basis for the null space. The dimension of the null space is based on the number of free variables. The number of columns minus the number of pivots.

$$Dim(Col\ A) + dim(Nul\ A) = n$$

n is the number of columns in the matrix

Row Space is the span of the rows of a matrix (these are all of dimension n). The dimension of the row space is the same as the number of pivots.

Rank of a matrix is the dimension of the column space (or the row space) (or the number of pivots).

$$Rank\ A + dim(Nul\ A) = n$$

Equivalencies to add to the Invertible matrix theorem:
m. the columns of A form a basis for R^n
n. $Col\ A = R^n$
o. dim Col A = n
p. Rank A = n
q. Nul A = {0} (on the trivial solution, the zero vector)
r. dim Nul A = 0

nullity = Dim(Nul A)

Change of Basis from the standard basis to a different (and back again). What if I have two non-standard bases that I want to switch between?

Non-standard basis #1 : B
Non-standard basis #2 : C

$$P_B[x]_B = x$$
$$P_C[x]_C = x$$

$$P_B[x]_B = P_C[x]_C$$

$$P_C^{-1}P_B[x]_B = [x]_C$$

Transforms a vector from the B basis to the coordinate representation in the C basis.

$$P_B^{-1}P_C[x]_C = [x]_B$$

Transforms a vector from the C basis to the B basis.

$$P_C^{-1}P_B = P_{C\leftarrow B}$$

Transformation from B to C (see arrow)

$$P_B^{-1}P_C = P_{B\leftarrow C}$$

Transformation from C to B (see arrow)

Like solving for inverses with row reducing. $[P_C|P_B] \rightarrow [I|P_{C\leftarrow B}]$

(I strongly recommend using the calculator).