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Vectors and Vector Spaces

Vectors and properties of vectors.

Vector space is a set V of vectors (or other object) that satisfies each of the following properties, if u and v and w are in the set V, c and d are real numbers.

1. $u + v$ is in the set V (closed under addition)
2. $u + v = v + u$
3. $(u + v) + w = u + (v + w)$
4. There is a zero vector in V such that $u + 0 = u$
5. For each u in V, there is a vector $-u$ such that $u + (-u) = 0$.
6. The scalar multiple of u by c is denoted by cu and is also in V. (closed under scalar multiplication)
7. $c(u + v) = cu + cv$
8. $(c + d)u = cu + du$
9. $(cd)u = c(du)$
10. $1u = u$ (identity)

Examples of vector space: R^n

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Is the set of polynomials (of a particular degree) P_n.

$$P_2 : at^2 + bt + c$$

Matrices are "vectors" in this sense.

$$0u = 0$$
$$c0 = 0$$
$$-u = (-1)u$$

Subset of a vector space may be a subspace (vector space), if V (the set the subset is drawn from) is a Vector Space, and the subset S satisfies the following properties:

1. Is the 0 vector in the subset S?
2. Is S closed under addition. If u, v are in S, then $u + v$ are also in S.
3. Is S closed under scalar multiplication? If u is in S, and c is any real number, is cu in S?

Let V be the first quadrant in xy-plane, i.e. $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, x \geq 0, y \geq 0 \right\}$

1. Yes, the zero vector is in the set.
2. $\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} z \\ w \end{bmatrix} = \begin{bmatrix} x + z \\ y + w \end{bmatrix}$ so this is good, this is in the set
3. $k \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} kx \\ ky \end{bmatrix}$, ex. $(-1) \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \end{bmatrix}$ is not in the set V.

This is not a subspace.

Let V be the set $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix}$. This is a subspace of R^3?

1. Is 0 in the set? Let $s = 0$, then $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Ok
2. $\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix} + \begin{bmatrix} t \\ 3t \\ 2t \end{bmatrix} = \begin{bmatrix} (s+t) \\ 3s+3t \\ 2s+2t \end{bmatrix} = \begin{bmatrix} (s+t) \\ 3(s+t) \\ 2(s+t) \end{bmatrix}$ this is in the set, closed under addition
3. $k\begin{bmatrix} s \\ 3s \\ 2s \end{bmatrix} = \begin{bmatrix} ks \\ 3ks \\ 2ks \end{bmatrix} = \begin{bmatrix} (ks) \\ 3(ks) \\ 2(ks) \end{bmatrix}$ this is in the set, closed under scalar multiplication

All polynomials of the form $p(t) = a + t^2$.
Is this a subspace? No.

What about $p(t) = at^2$? This is a subspace. $at^2 + bt^2 = (a+b)t^2$ is okay. $kat^2 = (ka)t^2$ (this is a subspace)

$$R^1 = [a]$$
$$R^2 = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$R^3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$P_n$$
$$P_0 = a(1)$$
$$P_1 = at + b$$
$$P_2 = at^2 + bt + c$$

Null spaces, Column Spaces, and Linear Transformations

The null space of a linear transformation (Nul A) is the set of all solution for the homogeneous equation $Ax = 0$.

In other words, Nul A is the set of dependent solutions to the homogeneous system.

Nul A always contains the zero vector. The number of free variables determines the "size" (dimension) of the null space.

Sometimes the null space is referred to as the kernel of the transformation.

Suppose we start with a transformation already in reduced echelon form:

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced $Ax = 0$ matrix.

$$\begin{gathered} x_1 - 2x_2 - x_4 + 3x_5 = 0 \\ x_2 = x_2 \\ x_3 + 2x_4 - 2x_5 = 0 \\ x_4 = x_4 \\ x_5 = x_5 \end{gathered}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} x_4 \\ 0 \\ -2x_4 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_5 \\ 0 \\ 2x_5 \\ 0 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} x_4 + \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} x_5$$

The Null space of a transformation A is a subspace of the DOMAIN. The Domain here is R^5.

Column Space of a matrix is a subset of codomain = range. Col A = $span\{a_1, a_2, \dots, a_n\}$ where a_i is a column of the A matrix. Col A = range of A.

This transformation is not onto and so does not span all of R^3 (does not have a pivot in every row).

Linearly independent sets. (pivots in every column)
Basis for a space is a set of vectors with two properties: 1) the vectors are linearly independent, 2) span the entire space (onto the whole space).

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

If the set has both independence and span, then the set is a basis.

The Null space is already (found the usual way) a set of independent vectors.
As long as we found all the vectors that define the null space, then that is a basis for the null space.

Column space is already going to span the column space (because of the definition). But, we have to pull only the vectors where a pivot occurs in order to state a basis for the column space.
(but we want the vectors from the original matrix A, not the reduced matrix).

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Reduces to this:

$$\begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Basis for Col A = $\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$

Spanning set Theorem:

If you have set that span a vector space (or subspace), then you can eliminate any vectors from the set that are dependent on other vectors in the set to make a basis.