8/26/2021

Linear algebra is the study of systems of linear equations.

A general linear equation:

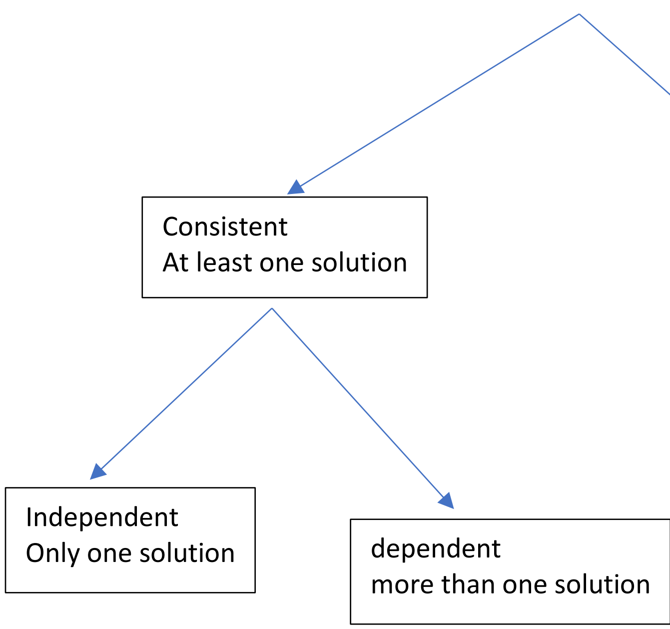
$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = a_0$$

More than one of these equations, then we have a system.

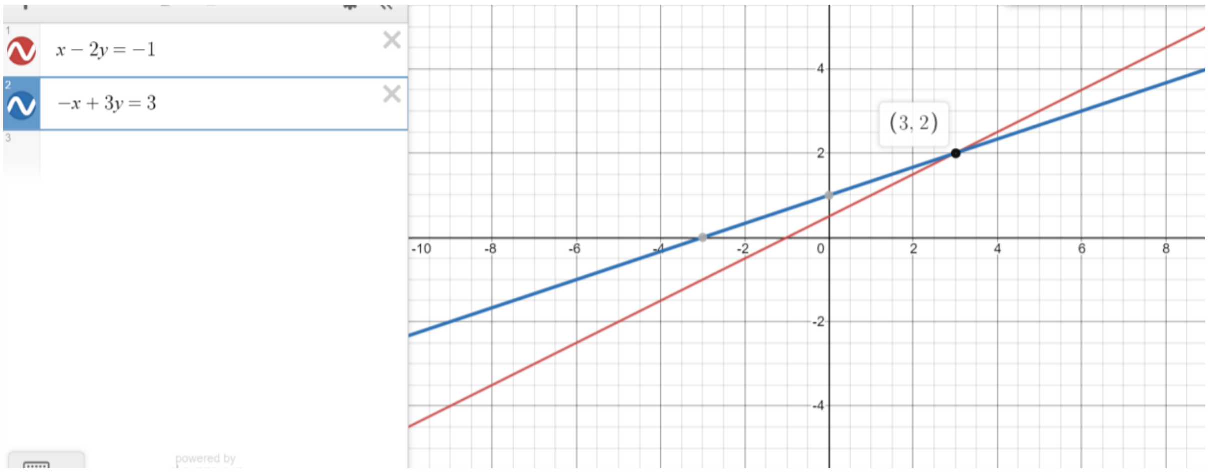
Systems that are overdetermined: the system cannot be solved because there are more equations than variables.

Systems that are underdetermined: they systems has too few equations to solve for a single point (fewer equations than variables).

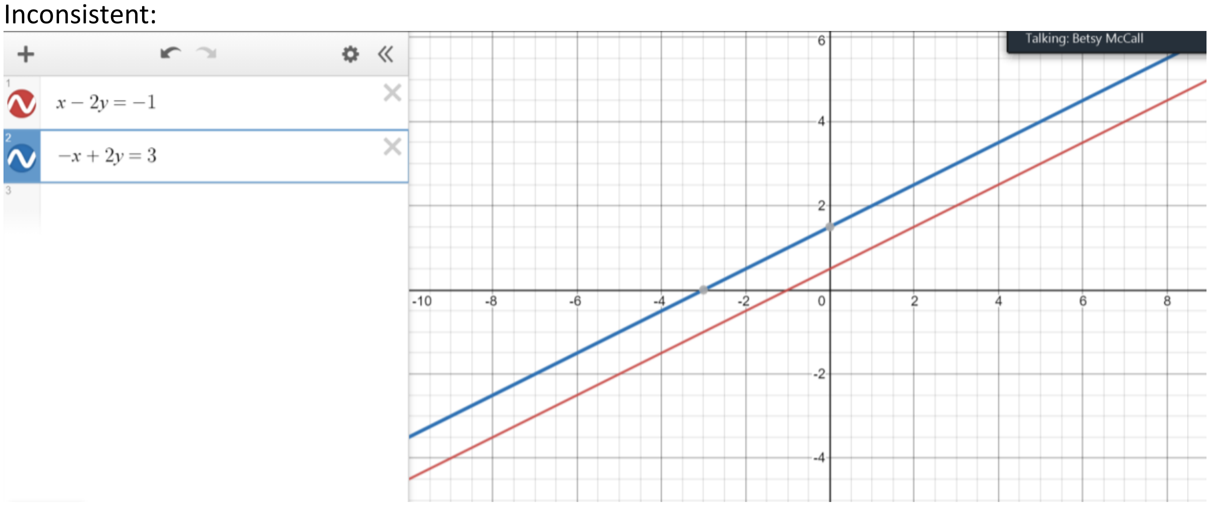
These undetermined systems are called dependent solutions.
Systems that can be solved (dependent or otherwise) are consistent. Consistent systems have at least one solution.
Independent solutions have only a single point (one value for each variable).
Inconsistent systems have no solution.



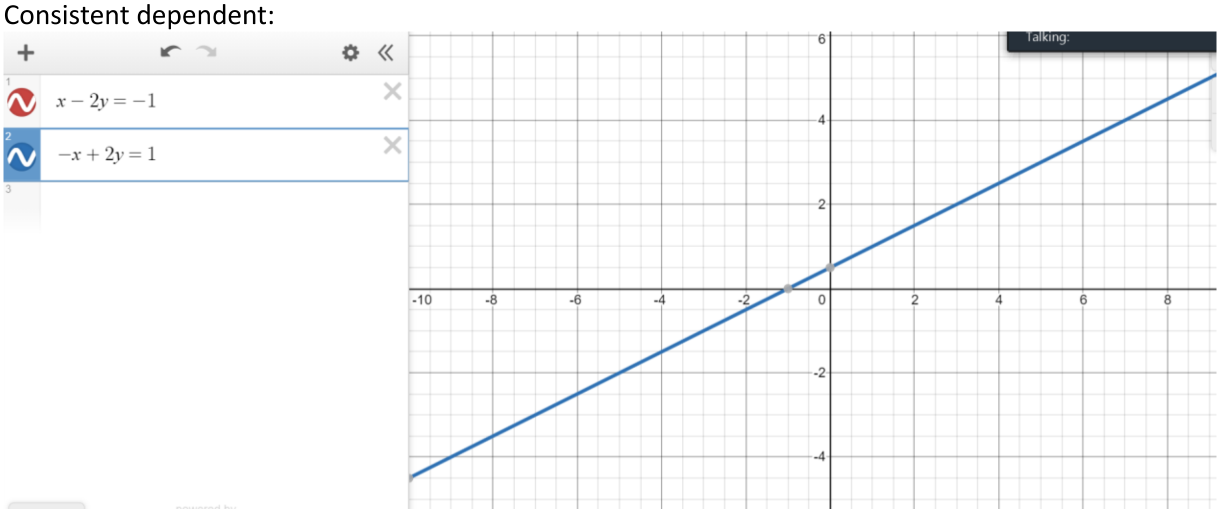
https://www.desmos.com/calculator
consistent independent:



Inconsistent:



Consistent dependent:



In previous courses, our methods for solving systems of equations (other than graphing):
Substitution
Elimination by addition

System:

$$x_1 - 2x_2 = -1$$
$$-x_1 + 3x_2 = 3$$

Using elimination: add the two equations together to eliminate x_1.

$$x_2 = 2$$

Once I have one variable, then plug in either equation to get the other.

$$x_1 - 2(2) = -1$$
$$x_1 = 3$$

Solution $(3,2)$

To solve as a matrix, we need to convert system into an augmented matrix.

$$\begin{bmatrix} 1 & -2 & -1 \\ -1 & 3 & 3 \end{bmatrix} or \left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$$

The portion of the matrix from the coefficients is called coefficient matrix. When there is a bar (or implied) the last column is called the augment.

To solve the system, we use Gaussian elimination, or Gauss-Jordan Elimination.

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ -1 & 3 & 3 \end{array}\right]$$

We want to add rows together so that one of the coefficients cancels (it becomes 0)
Just like with elimination by addition, we can multiply rows by something if needed to get the coefficients to match.

The third thing you can do is exchange rows. (Generally, you only have to do this if coefficient at top left is zero).

Take the top-left position and use it to make everything under it a zero. The top left position should also be a 1. (pivot)

$$R_1 + R_2 \to R_2$$

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

Next: go over a column and down a row (2nd row, 2nd column). Make this position a one (pivot), and then use that position to eliminate coefficients beneath it.

This is in echelon form.

Can use to backsolve from here by pulling the equations back out of the matrix.

$$x_1 - 2x_2 = -1$$

$$x_2 = 2$$

Reduced (row) echelon form: once you finish the echelon form and find all pivots, then use the pivots (starting on the right) to work your way back up: eliminate all the coefficients above the pivots as well.

$$\left[\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 1 & 2 \end{array}\right]$$

$$2R_2 + R_1 \to R_1$$

$$\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right]$$

$$\begin{aligned} x_1 &= 3 \\ x_2 &= 2 \end{aligned}$$

Vector form of the solution $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Three variables:

$$\begin{cases} x_1 - 2x_2 + x_3 = 0 \\ \quad\ \ 2x_2 - 8x_3 = 8 \\ 5x_1 \qquad\ - 5x_3 = 10 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 5 & 0 & -5 & 10 \end{array}\right]$$

$$-5R_1 + R_3 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & 10 & -10 & 10 \end{array}\right]$$

$$\frac{1}{2}R_2 \to R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 10 & -10 & 10 \end{array}\right]$$

$$-10R_2 + R_3 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 30 & -30 \end{array}\right]$$

$$\frac{1}{30}R_3 \to R_3$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Echelon form.

$$x_1 - 2x_2 + x_3 = 0$$
$$x_2 - 4x_3 = 4$$
$$x_3 = -1$$

$$x_2 - 4(-1) = 4$$
$$x_2 = 0$$

$$x_1 - 2(0) + (-1) = 0$$
$$x_1 = 1$$

Solution $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$4R_3 + R_2 \to R_2$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$-R_3 + R_1 \to R_1$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

$$2R_2 + R_1 \to R_1$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Reduced (row) echelon form, the last column is the solution.

How to do in calculator.

Size of a matrix is given by the rows (equations), then columns (variables+constants). 3×4

What is echelon form generally for large matrices?

For large matrices, pivots may disappear (or move right).

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \end{array}\right]$$

Reduced echelon form can only have * to the right of the last pivot column. Everything else is 0 or 1.

Vector Equations and Matrix Equations

A vector equation is of the form:

$$x_1 \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ -8 \\ -5 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 10 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Matrix equation:

$$Ax = b$$

$$\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ 5 & 0 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 10 \end{bmatrix}$$

These notes cover 1.1-1.4 in Lay.