9/2/2021

1d. (homework)

$$\begin{cases} 2x_1 - 6x_3 = -8 \\ x_2 + 2x_3 = 3 \\ 3x_1 + 6x_2 - 2x_3 = -4 \end{cases}$$

$$\begin{bmatrix} 2 & 0 & -6 & -8 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{bmatrix}$$

$$\frac{1}{2}R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 3 & 6 & -2 & -4 \end{bmatrix}$$

$$-3R_1 + R_3 \to R_3$$

$$\begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 6 & 7 & 8 \end{bmatrix}$$

$$-6R_2 + R_3 \to R_3$$

$$\begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$

$$-\frac{1}{5}R_3 \to R_3$$

$$\begin{bmatrix} 1 & 0 & -3 & -4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$-2R_3 + R_2 \to R_2$$
$$3R_3 + R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$x = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

Lay Section 1.5

Homogeneous systems
Homogenous system has all the constants (in the augment) are zero

Homogeneous systems always have at least one solution: when all the variables are zero. This solution is called the **trivial** solution, because every homogeneous system has this as a solution. Homogeneous systems are always consistent.

The question is whether the system has only one solution (the trivial solution), or whether it has more than one solution, does it have a non-zero solution. What we are looking is the dependent solution.

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 0 \\ -3x_1 - 2x_2 + 4x_3 = 0 \\ 6x_1 + x_2 - 8x_3 = 0 \end{cases}$$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ -3 & -2 & 4 & 0 \\ 6 & 1 & -8 & 0 \end{bmatrix}$$

$$R_1 + R_2 \rightarrow R_2$$
$$-2R_1 + R_3 \rightarrow R_3$$

$$\begin{bmatrix} 3 & 5 & -4 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{3}R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & \frac{5}{3} & -\frac{4}{3} & \frac{0}{3} \\ 0 & 3 & 0 & 0 \\ 0 & -9 & 0 & 0 \end{bmatrix}$$

$$3R_2 + R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & \frac{5}{3} & -\frac{4}{3} & \frac{0}{3} \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\frac{1}{3}R_2 \rightarrow R_2$$

$$\begin{bmatrix} 1 & \frac{5}{3} & -\frac{4}{3} & \frac{0}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Homogeneous systems can’t be backsolved. Must go all the way to **reduced** echelon form.

$$-\frac{5}{3}R_2 + R_1 \rightarrow R_1$$

$$\begin{bmatrix} 1 & 0 & -\frac{4}{3} & \frac{0}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In reduced echelon form: convert back to a system of equations.

$$x_1 - \frac{4}{3}x_3 = 0$$
$$x_2 = 0$$

x_3 is a free variable. $x_1 = \frac{4}{3}x_3$

$$x_1 = \frac{4}{3}x_3$$
$$x_2 = 0$$
$$x_3 = x_3$$

Equivalent way:

$$x_1 = \frac{4}{3}t$$
$$x_2 = 0$$
$$x_3 = t$$

$$x = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} t = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} s$$

Parametric solution
If we let $x_3 = 3s$

If we had a system that was not homogeneous, but was otherwise the same, the solution to that system, would be the same solution but with a shift.

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 7 \\ -3x_1 - 2x_2 + 4x_3 = -1 \\ 6x_1 + x_2 - 8x_3 = -4 \end{cases}$$

$$\begin{bmatrix} 1 & 0 & -4/3 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 - \frac{4}{3}x_3 = -1$$
$$x_2 = 2$$

$$x_3 = x_3$$

$$x_1 = \frac{4}{3}x_3 - 1$$
$$x_2 = 2$$
$$x_3 = x_3$$

$$x = \begin{bmatrix} \frac{4}{3} \\ 0 \\ 1 \end{bmatrix} x_3 + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} t + \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

Parametric solution

$x = tv$ as the solution to the homogeneous system, then $x = tv + p$ is the solution to the non-homogeneous system (if it exists).

$$\begin{cases} 3x_1 + 5x_2 - 4x_3 = 7 \\ -3x_1 - 2x_2 + 4x_3 = -1 \\ 6x_1 + x_2 - 8x_3 = 4 \end{cases}$$

$$\begin{bmatrix} 1 & 0 & -4/3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x_1 - \frac{4}{3}x_3 = 0$$
$$x_2 = 0$$
$$0 = 1$$

This system is inconsistent. It has no solution.

Independent systems have a pivot in every column except the augment. (Pivots in augments are inconsistent).
Dependent systems have a least one column in the coefficient matrix that is missing a pivot.

Linear Independence
Linear independence is a property of a set of vectors.
A set of vectors in R^n is said to be linearly independent if the vector equation $x_1v_1 + x_2v_2 + \cdots + x_nv_n = 0$ has only the trivial solution. It is linearly dependent if this same equation has more than the trivial solution.

The procedure for solving the system is to put the vectors into a matrix (can leave out the augment), and row-reduce to see if there is a pivot in every column of the coefficient matrix.

System/set of vectors:

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$x_1\begin{bmatrix}1\\2\\3\end{bmatrix}+x_2\begin{bmatrix}4\\5\\6\end{bmatrix}+x_3\begin{bmatrix}2\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Is the solution trivial only, or is it dependent?

$$\begin{aligned}x_1+4x_2+2x_3&=0\\2x_1+5x_2+x_3&=0\\3x_1+6x_2&=0\end{aligned}$$

$$\begin{bmatrix}1&4&2&0\\2&5&1&0\\3&6&0&0\end{bmatrix}$$

Essentially, we've put the vectors into the columns of a matrix. Row-reduce to look for pivots in every column. (The solution to the system independent… and the vectors linearly independent.)

The system reduces to

$$\begin{bmatrix}1&2&0\\0&1&1\\0&0&0\end{bmatrix}$$

This system is dependent, and so the vectors are dependent.
If there is a pivot in every column (where there was an original vector) the vectors are independent.

Telling if vectors are independent/dependent without the matrix methods:

If you have two vectors, they are not multiples of each other, then they are independent.
(You can't do this if there are three or more.)

Any set of vectors that contains the 0 vector is dependent

Some sets of vectors will already be in "echelon form".

$$\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix}$$

If there are more vectors than components of the vector, then the set is dependent.

$$\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\4\end{bmatrix},\begin{bmatrix}1\\1\end{bmatrix}$$

The matrix doesn't have enough rows to have a pivot in every column, so dependent.

Span of a set of vectors: is the set of linear combinations of the vectors in the set.

$$\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix}$$

What's in the span? $span\left\{\begin{bmatrix}1\\0\\0\end{bmatrix},\begin{bmatrix}2\\3\\0\end{bmatrix},\begin{bmatrix}1\\2\\3\end{bmatrix}\right\}$

Each vector is in the span. Eg. $\begin{bmatrix}1\\0\\0\end{bmatrix}$

Multiples of each vector are in span. Eg. $k\begin{bmatrix}2\\3\\0\end{bmatrix}$

Sums of vectors from the set are in span, eg. $\begin{bmatrix}1\\0\\0\end{bmatrix}+\begin{bmatrix}2\\3\\0\end{bmatrix}=\begin{bmatrix}3\\3\\0\end{bmatrix}$

Add scalar multiples of these vectors, they are in the span, eg. $2\begin{bmatrix}1\\0\\0\end{bmatrix}-3\begin{bmatrix}2\\3\\0\end{bmatrix}+4\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}0\\-1\\12\end{bmatrix}$

Definition does not depend on the vectors being independent.

Proofs

You want to begin with the assumptions given in the proof (if statements), and from those, logically follow steps without adding new assumptions, to arrive at the conclusion of the proof (then statement).

If T is a linear transformation, then $T(0)=0$, and $T(cu+dv)=cT(u)+dT(v)$.

Go back to the definition of a linear transformation. What is it? What properties must be true if this is a linear transformation. Work from those properties to prove the then statement.

If p then q $\leftrightarrow$ if not q, then not p (contrapositive)

Proof by contradiction: assume the if statement is true and the conclusion is false, show that this produces a contradiction: that it's impossible for that to be the case.

https://www.vcccd.edu/sites/default/files/departments/human-resources/sabbaticals/2017-2018/paulcurtis-sabbaticalfinalrpt-fall2017-proofsinlinearalgebra.pdf

https://people.engr.tamu.edu/pcr/courses/csce222/spring15/lectures/03_Proof_Methods_and_Strategies.pdf

Plus, my own handout. Can find more examples by searching for linear algebra proofs.

Quiz due Saturday.

$$\begin{bmatrix}1 & 4 & 2 & 0\\2 & 5 & 1 & 0\\3 & 6 & 0 & 0\end{bmatrix}$$

$$-2R_1+R_2\rightarrow R_2$$

$$-3R_1 + R_3 \rightarrow R_3$$

$$\begin{bmatrix} 1 & 4 & 2 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & -6 & -6 & 0 \end{bmatrix}$$