9/9/2021

e. $\begin{bmatrix} 1 & h & -5 \\ 2 & -8 & 6 \end{bmatrix}$, consistent

$$-2R_1 + R_2 \to R_2$$

$$\begin{bmatrix} 1 & h & -5 \\ 0 & -2h-8 & 16 \end{bmatrix}$$
$$-2h - 8 \neq 0$$

a. $\begin{bmatrix} 1 & 2 & 4 & 5 \\ 2 & 4 & 5 & 4 \\ 4 & 5 & 4 & 2 \end{bmatrix}$

$$-2R_1 + R_2 \to R_2, -4R_1 + R_3 \to R_3$$

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 0 & -3 & -6 \\ 0 & -3 & -12 & -18 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & -3 & -12 & -18 \\ 0 & 0 & -3 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & -3 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 4 & 5 \\ 0 & 1 & 4 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$-4R_3 + R_2 \to R_2$$
$$-4R_3 + R_1 \to R_1$$

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

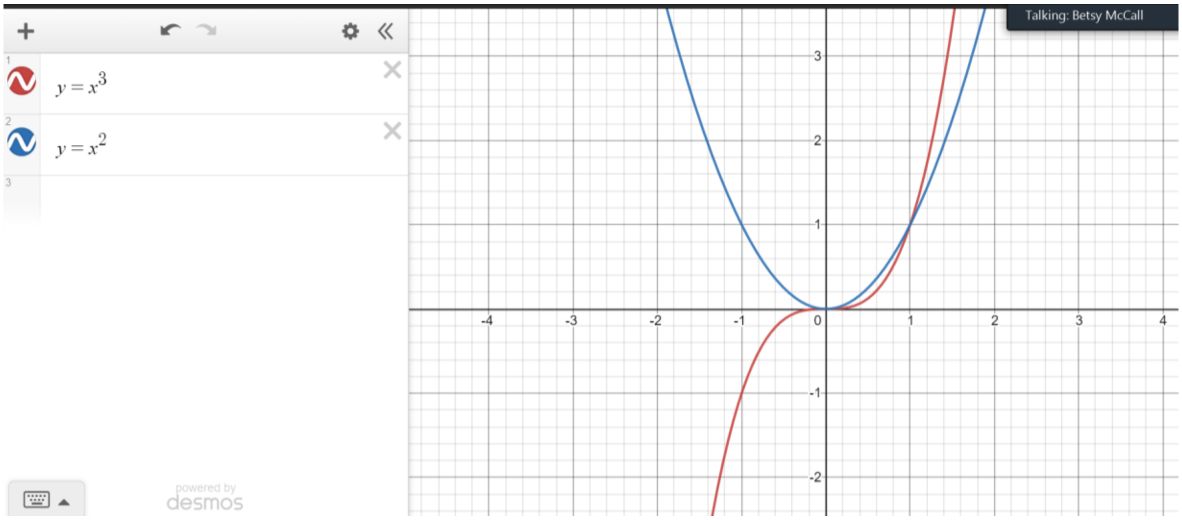
$$-2R_2 + R_1 \to R_1$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Linear Transformations

A transformation is a function (a mapping) T from R^n to R^m is a rule that assigns each vector in R^n to a vector in R^m. The set R^n that is the input to the transformation is called the domain. The set R^m is the set the transformation is mapping into, and that is called the codomain.

There is a difference between the codomain and the range.



$$f(x) = x^3, g(x) = x^2$$

If a function is onto, then the range and the codomain are equal.
If a function is one-to-one, then every vector from the domain maps onto a unique vector in the range (no two vectors map onto the same value in the range).

Linear transformation

Properties that must be satisfied to be linear:

1) $T(0) = 0$ (functions that are "linear" otherwise but misses this property are called affine transformations)
2) $T(u) + T(v) = T(u + v)$ -- closed under addition
3) $kT(u) = T(ku)$ – closed under scalar multiplication

If we want to prove that a transformation is linear, then we have to prove these three properties.

Linear transformations can be expressed as matrices, but they don't have to be.

$$T(x) = Ax$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_3 + 7x_1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 1 - 3(2) \\ 3(1) + 5(2) \\ -3 + 7(1) \end{bmatrix} = \begin{bmatrix} -5 \\ 13 \\ 4 \end{bmatrix}$$

$$T(x) = Ax: R^3 \mapsto R^3$$

Do our three properties work?

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 - 3(0) \\ 3(0) + 5(0) \\ -0 + 7(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

That first property checks out.

$$u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

$$T(u) + T(v) = T(u + v)$$

$$T(u) = T\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 - 3u_2 \\ 3u_1 + 5u_2 \\ -u_3 + 7u_1 \end{bmatrix}, T(v) = T\left(\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = \begin{bmatrix} v_1 - 3v_2 \\ 3v_1 + 5v_2 \\ -v_3 + 7v_1 \end{bmatrix}$$

$$\begin{aligned} T(u) + T(v) &= \begin{bmatrix} u_1 - 3u_2 \\ 3u_1 + 5u_2 \\ -u_3 + 7u_1 \end{bmatrix} + \begin{bmatrix} v_1 - 3v_2 \\ 3v_1 + 5v_2 \\ -v_3 + 7v_1 \end{bmatrix} = \begin{bmatrix} u_1 - 3u_2 + v_1 - 3v_2 \\ 3u_1 + 5u_2 + 3v_1 + 5v_2 \\ -u_3 + 7u_1 - v_3 + 7v_1 \end{bmatrix} \\ &= \begin{bmatrix} u_1 + v_1 - 3(u_2 + v_2) \\ 3(u_1 + v_1) + 5(u_2 + v_2) \\ -(u_3 + v_3) + 7(u_1 + v_1) \end{bmatrix} \end{aligned}$$

$$T(u + v) = T\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}\right) = T\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + v_1 - 3(u_2 + v_2) \\ 3(u_1 + v_1) + 5(u_2 + v_2) \\ -(u_3 + v_3) + 7(u_1 + v_1) \end{bmatrix}$$

Satisfies the addition property

What about $kT(u) = T(ku)$?

$$kT(u) = kT\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = k\begin{bmatrix} u_1 - 3u_2 \\ 3u_1 + 5u_2 \\ -u_3 + 7u_1 \end{bmatrix} = \begin{bmatrix} k(u_1 - 3u_2) \\ k(3u_1 + 5u_2) \\ k(-u_3 + 7u_1) \end{bmatrix}$$

$$T(ku) = T\left(\begin{bmatrix} ku_1 \\ ku_2 \\ ku_3 \end{bmatrix}\right) = \begin{bmatrix} ku_1 - 3ku_2 \\ 3ku_1 + 5ku_2 \\ -ku_3 + 7ku_1 \end{bmatrix}$$

Does satisfy the scalar multiplication property.
All the required properties are satisfied, so the transformation is linear.

What makes a transformation nonlinear? Any non-linear operation (squares, square roots, division, multiplying by another variable, a loose constant, etc.)

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + |x_2| \\ x_2^2 \\ x_2 + 7 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 7 \end{bmatrix}$$

$$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 + |x_2| \\ x_2^2 \end{bmatrix}$$

You just need to find one counterexample to show that at least one property fails.

$$T\left(\begin{bmatrix} -2 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$$T\left(\begin{bmatrix} -3 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 16 \end{bmatrix}$$

$$T\left(\begin{bmatrix} -2 \\ -2 \end{bmatrix}\right) + T\left(\begin{bmatrix} -3 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 20 \end{bmatrix}$$

$$T\left(\begin{bmatrix} -2 \\ -2 \end{bmatrix} + \begin{bmatrix} -3 \\ -4 \end{bmatrix}\right) = T\left(\begin{bmatrix} -5 \\ -6 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 36 \end{bmatrix}$$

These are not equal, so the property fails.

$$(-1)T\left(\begin{bmatrix} -2 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

The scalar multiplication property is not satisfied.

Representing transformations as matrices

Rotation Matrix for a vector (positive angles are counterclockwise)

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Rotate counterclockwise 45-degrees, $\frac{\pi}{4}$ radians

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Reflection across x_1 $(x - axis)$ $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (vertical reflection)

Reflection across the x_2 (y-axis) $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ (horizontal reflection)

Reflection across the$y = x$ line $(x_1 = x_2)$ $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ (switches x and y)

Reflection across the$-x_1 = x_2$ (y=-x) $\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$ (both switches x and y, and changes the sign)

Reflection across the origin. $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ (switches signs but not positions)

Horizontal stretch/compression $\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$ (stretches x_1 by k)

Vertical stretch/compression $\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$ (stretches x_2 by k)

Whole vector (like scalar multiplication) $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

Shears

Horizontal shear $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$

Vertical shear $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

Projection transformations

Horizontal projection $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

Vertical projection $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Successive transformations work like composition function.

T=reflection across y=x

S=shear

P=projection

$$T(x) = Ax$$
$$S(x) = Bx$$
$$P(x) = Cx$$

Reflection, then shear : $T(x) = Ax \longrightarrow S(T(x)) = S(Ax) = B(Ax) = BAx$

Then do the projections $\longrightarrow P\left(S(T(x))\right) = P(BAx) = C(BAx) = CBAx$

CBA matrix would be the result of all three transformations applied in this order to the vector x

A function is onto R^m (codomain) is for each vector b in R^m, it is the image of a vector in R^n (domain) Image=output of the transformation for the given vector: from a matrix perspective, there need to be m pivots (one pivot in every row of the reduced matrix)

A function is one-to-one (image is unique in the domain=one way to get to each vector in the range), if each vector in the codomain is the image of exactly one vector in the domain. From a matrix perspective: there needs to be n pivots (one pivot in every column) (no free variables) – homogeneous system has only the trivial solution.

Systems that are square (the same number of rows as variables), then the transformation will be either both or neither.

$$e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

A transformation maps from $R^3 \mapsto R^4$, by $T(e_1) = \begin{bmatrix}4\\3\\2\\0\end{bmatrix}, T(e_2) = \begin{bmatrix}1\\-1\\1\\6\end{bmatrix}, T(e_3) = \begin{bmatrix}4\\0\\-2\\3\end{bmatrix}$

Write matrix of the transformation.

$$\begin{bmatrix}4 & 1 & 4\\3 & -1 & 0\\2 & 1 & -2\\0 & 6 & 3\end{bmatrix}$$

$$T\left(\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}\right) = \begin{bmatrix}x_1 - 3x_2\\3x_1 + 5x_2\\-x_3 + 7x_1\end{bmatrix}$$

$$\begin{bmatrix}1 & -3 & 0\\3 & 5 & 0\\7 & 0 & -1\end{bmatrix}$$

Chapter 2 starts (might swing back to talk about 1.10)

Matrices as objects – as vectors – defined by number of rows and the number of columns: $m \times n$. Perform addition and scalar multiplication component by component

$$2\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix} = \begin{bmatrix}4 & 6\\8 & 10\end{bmatrix}$$

Add matrices together if they are exactly the same size

$$\begin{bmatrix}2 & 3\\4 & 5\end{bmatrix} + \begin{bmatrix}1 & 2\\0 & -3\end{bmatrix} = \begin{bmatrix}3 & 5\\4 & 2\end{bmatrix}$$

To prove these, you can only use the definition of addition, scalar multiplication, and properties of real numbers inside the components.

Commutative property of matrices

$$A + B = B + A$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} \\ a_{21} + b_{21} & a_{22} + b_{22} \end{bmatrix}$$

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} + \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11} + a_{11} & b_{12} + a_{12} \\ b_{21} + a_{21} & b_{22} + a_{22} \end{bmatrix}$$

Equal by commutative property of real numbers

Matrix multiplication

Is defined for an $m \times n$ matrix multiplied by an $n \times p$ matrix, and the result is an $m \times p$ matrix

The number of columns in the left matrix have to match the number of rows in the right matrix

$2 \times 2\ \ times\ 2 \times 2$ is okay

$2 \times 3\ times\ 2 \times 3$ is not okay – undefined

$2 \times 3\ times\ 3 \times 2$ – result would be a 2×2.

$$\begin{bmatrix} 1 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1(3) + 2(2) \\ 4(3) + (-2)(2) \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 3 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1(1) + 2(3) & 1(0) + 2(4) & 1(1) + 2(2) \\ 4(1) - 2(3) & 4(0) - 2(4) & 4(1) - 2(2) \end{bmatrix} = \begin{bmatrix} 7 & 8 & 5 \\ -2 & -8 & 0 \end{bmatrix}$$

$$AB \neq BA$$

$\mathrm{I_n}$ identity matrix

Transposing : switching the rows into the columns.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$(BA)^T = A^T B^T$$