Math 266, Quiz #12, Fall 2021 Name KEY

Instructions: Show all work. Some problems will instruct you to complete operations by hand, some can be done in the calculator. To show work on calculator problems, show the commands you used, and the resulting matrices. **Give exact answers** (yes, that means fractions, square roots and exponentials, and not decimals) unless specifically directed to give a decimal answer. This will require some operations to be done by hand even if not specifically directed to. Be sure to complete all parts of each question.

1. Find the similarity transformation, if it exists, of the matrix $A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 5 & -2 \\ 1 & 3 & 1 \end{bmatrix}$ that is capable of diagonalizing the matrix.

$$\left|\begin{bmatrix} 1-\lambda & 2 & -3 \\ 2 & 5-\lambda & -2 \\ 1 & 3 & 1-\lambda \end{bmatrix}\right| = (1-\lambda)\begin{bmatrix} 5-\lambda & -2 \\ 3 & 1-\lambda \end{bmatrix} - 2\begin{bmatrix} 2 & -2 \\ 1 & 1-\lambda \end{bmatrix} - 3\begin{bmatrix} 2 & 5-\lambda \\ 1 & 3 \end{bmatrix} =$$

$$(1-\lambda)\left[(5-\lambda)(1-\lambda)+6\right] - 2\left(2(1-\lambda)+2\right) - 3\left(6-(5-\lambda)\right) =$$

$$(1-\lambda)\left[5-6\lambda+\lambda^2+6\right] - 2\left[2-2\lambda+2\right] - 3\left[6-5+\lambda\right] =$$

$$(1-\lambda)\left[\lambda^2-6\lambda+11\right] - 2\left[4-2\lambda\right] - 3\left[1+\lambda\right] =$$

$$\lambda^2 - 6\lambda + 11 - \lambda^3 + 6\lambda^2 - 11\lambda - 8 + 4\lambda - 3 - 3\lambda =$$

$$-\lambda^3 + 7\lambda^2 - 16\lambda = 0$$

$$-\lambda(\lambda^2 - 7\lambda + 16) = 0$$

$$\lambda = 0 \qquad \lambda = \frac{7 \pm \sqrt{49-64}}{2} = \frac{7 \pm \sqrt{15}\,i}{2}$$

matrix cannot be diagonalized in real #'s

2. Find the similarity transformation for the matrix $B = \begin{bmatrix} 0 & 5 \\ -2 & 2 \end{bmatrix}$ that converts this matrix into a similar rotation matrix. Then use that matrix to find the angle of rotation.

$-\lambda(2-\lambda) + 10 =$

$\lambda^2 - 2\lambda + 10 = 0$

$\lambda = \frac{2 \pm \sqrt{4-40}}{2} = \frac{2 \pm 6i}{2} = 1 \pm 3i$

$\lambda = 1 - 3i$

$A = \begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$ $\quad \theta = \cos^{-1}\left(\frac{1}{\sqrt{10}}\right) = 1.249$ radians

71.57°

$\begin{bmatrix} -1+3i & 5 \\ -2 & 1+3i \end{bmatrix}$

$(-2)x_1 + (1+3i)x_2 = 0$

$x_1 = \frac{1+3i}{2} x_2$

$\vec{v}_2 = \begin{bmatrix} 1+3i \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \end{bmatrix} i$

$C = \begin{bmatrix} 1 & 3 \\ 2 & 0 \end{bmatrix}$

For the 4 problems below (3-6) select two (2) problems to complete. You may do the others for extra credit if you wish.

3. Find the equilibrium vector of the matrix $P = \begin{bmatrix} .1 & .5 \\ .9 & .5 \end{bmatrix}$ algebraically. Be sure to properly normalize the vector.

$(.1-\lambda)(.5-\lambda) - .45 = 0$

$\lambda^2 - .6\lambda - .05 - .45 = 0$

$\lambda^2 - .6\lambda - .5 = 0$

$\lambda = -.4$

$\lambda = 1$

$\lambda = 1$

$\begin{bmatrix} -.9 & .5 \\ .9 & -.5 \end{bmatrix}$

$.9x_1 = .5x_2$

$x_1 = \frac{9}{5} x_2$

$\vec{v} = \begin{bmatrix} 9 \\ 5 \end{bmatrix}$ $\quad 9 + 5 = 14$

$\vec{x} = \begin{bmatrix} 9/14 \\ 5/14 \end{bmatrix}$

4. Use your calculator to find the equilibrium vector of the stochastic matrix $P = \begin{bmatrix} .7 & .1 & .1 \\ .2 & .8 & .2 \\ .1 & .1 & .7 \end{bmatrix}$.

Explain the steps you took to obtain the vector. Then demonstrate that it is the correct equilibrium vector by multiplying by P.

$\lim_{n \to \infty} P^n = \begin{bmatrix} .25 & .25 & .25 \\ .5 & .5 & .5 \\ .25 & .25 & .25 \end{bmatrix}$

$\vec{x} = \begin{bmatrix} .25 \\ .5 \\ .25 \end{bmatrix} = \begin{bmatrix} 1/4 \\ 1/2 \\ 1/4 \end{bmatrix}$

$Px = \vec{x}$

5. Solve the discrete dynamical system given by $\vec{x}_{k+1} = \begin{bmatrix} 0.3 & 0.4 \\ -0.3 & 1.1 \end{bmatrix} \vec{x}_k$. Sketch a graph of the eigenvectors and plot some sample trajectories. Is the origin an attractor, a repeller or a saddle point?

$(.3-\lambda)(1.1-\lambda)+.12=0$

$\lambda^2-1.4\lambda+.45=0$

$\lambda=0.5 \quad \lambda=0.9$

$\begin{bmatrix} -.2 & .4 \\ -.3 & .6 \end{bmatrix}$

$-.2x_1=-.4x_2$

$x_1=2x_2$
$x_2=x_2$

$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

0.5^k

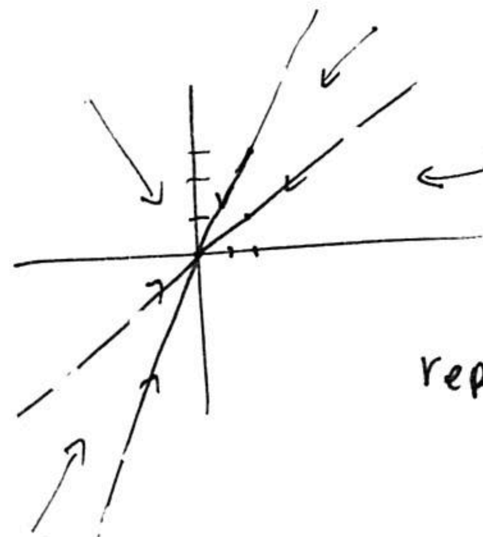
$\begin{bmatrix} -.6 & .4 \\ -.9 & .2 \end{bmatrix}$

$-.6x_1=-.4x_2$

$x_1=\frac{2}{3}x_2$
$x_2=x_2$

$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

survives longer 0.9^k



repeller

trajectories all go toward the origin

drift toward $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ vector

6. Solve the system of ODEs given by $\vec{x}' = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix} \vec{x}$. Sketch a graph of the eigenvectors and plot some sample trajectories. Is the origin an attractor, a repeller or a saddle point?

$(2-\lambda)(-2-\lambda)+3=0$

$-4+\lambda^2+3=$
$\lambda^2-1=0$

$\lambda=\pm1$

$\lambda=1$ $\begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}$ grows

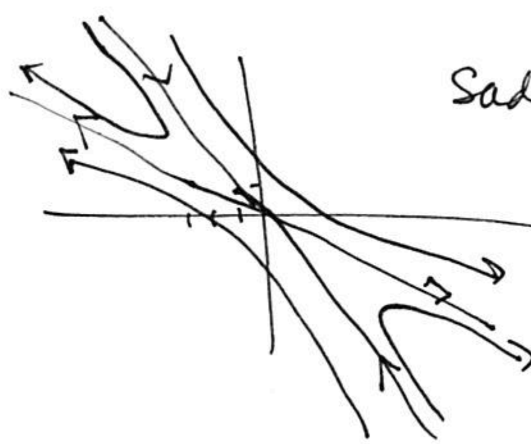
$x_1=-3x_2$
$x_2=x_2$

$v_1=\begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\lambda=-1$ $\begin{bmatrix} 3 & 3 \\ -1 & -1 \end{bmatrix}$ Shrinks

$x_1=-x_1$
$x_2=x_2$

$\bar{v}_2=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$



Saddle point