MTH 264, Exam #2, Fall 2022 Name KEY

Instructions: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error of any kind.

1. For each of the series below, determine whether the series converges or diverges. (6 points each)

   a. $\sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right)$

      $=\sum_{n=1}^{\infty} [\ln(n+1) - \ln(n)]$

      telescoping series

      diverges

      $\lim_{n\to\infty} \ln n = \infty$

   b. $\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$ n th term test diverges

      $\lim_{n\to\infty} \frac{n+1}{2n-1} = \frac{1}{2} \neq 0$

   c. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt[3]{(\ln n)^2}}$ integral test $\int_2^{\infty} \frac{1}{n(\ln n)^{2/3}}\,dn$ $u = \ln n$, $du = \frac{1}{n}$

      $\int_{\ln 2}^{\infty} u^{-2/3}\,du = 3u^{1/3}\Big|_{\ln 2}^{\infty} = \infty$

      diverges

   d. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n^2+2}}$ limit comparison to $\frac{1}{n}\cdot\frac{1}{\sqrt{3}}$

      $\lim_{n\to\infty} \frac{1/\sqrt{3n^2+2}}{\frac{1}{\sqrt{3}\,n}} = \lim_{n\to\infty} \frac{\sqrt{3}\cdot n}{\sqrt{3n^2+2}} = 1$ converges or diverges together.

      $\frac{1}{n}$ diverges by the p-test so this also diverges

   e. $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$ integral test

      $\int_1^{\infty} \tan\left(\frac{1}{n}\right)dn$ diverges

f. $\sum_{n=2}^{\infty}\frac{n}{(n^2-1)^2}$ limit Comparison w/ $\frac{1}{n^3}$ $\lim_{n\to\infty}\frac{\frac{n}{(n^2-1)^2}}{1/n^3} = \lim_{n\to\infty}\frac{n^4}{n^4-2n^2+1} = 1$

converges or diverges together

$\frac{1}{n^3}$ converges by p-test so this also converges

g. $\sum_{n=0}^{\infty}\frac{\cos n\pi}{n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+1}$ converges by the alternating series test

$\lim_{n\to\infty}\frac{1}{n+1} = 0$

h. $\sum_{n=1}^{\infty}\frac{(n+1)(-3)^n}{n!}$ ratio test $\lim_{n\to\infty}\frac{(n+2)(-3)^{n+1}}{(n+1)!}\cdot\frac{n!}{(n+1)(-3)^n} = \lim_{n\to\infty}\frac{(-3)(n+2)}{(n+1)(n+1)} = 0$

converges

i. $\sum_{n=0}^{\infty}\frac{(n!)^2}{(3n)!}$ ratio test $\lim_{n\to\infty}\frac{(n+1)!\,(n+1)!}{(3n+3)!}\cdot\frac{(3n)!}{n!\,n!} = \lim_{n\to\infty}\frac{(n+1)(n+1)}{(3n+1)(3n+2)(3n+3)} = 0$

converges

j. $\sum_{n=1}^{\infty}\frac{11}{n^{9/8}}$ converges by the p-series test

$p = 9/8 > 1$

2. Find N such that $R_N \leq 10^{-5}$, for the convergent series. (10 points)

33 terms $\quad\sum_{n=1}^{\infty}\frac{1}{n^4} = \pi^4/90 = 1.08232$ $\qquad \int_N^{\infty}\frac{1}{x^4}dx = \frac{-1}{3x^3}\Big|_N^{\infty} = \frac{1}{3N^3} = 10^{-5}$

$N^3 = 3\times10^5$

integral test says 67

$1+\frac{1}{16}+\frac{1}{81}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\frac{1}{7^4}+\frac{1}{8^4}+\frac{1}{9^4}+\frac{1}{10^4}+\frac{1}{11^4}+\frac{1}{12^4}+\frac{1}{13^4}+\frac{1}{14^4}+\frac{1}{15^4}+\frac{1}{16^4}+\frac{1}{17^4}+\frac{1}{18^4}$
$+\frac{1}{19^4}+\frac{1}{20^4}+\frac{1}{21^4}+\frac{1}{22^4}+\frac{1}{23^4}+\frac{1}{24^4}+\frac{1}{25^4}+\frac{1}{26^4}+\frac{1}{27^4}+\frac{1}{28^4}+\frac{1}{29^4}+\frac{1}{30^4}+\frac{1}{31^4}+\frac{1}{32^4}+\frac{1}{33^4}$

3. For the sequence below. i) Determine if the sequence is monotonic (or is monotonic after some finite value of n). You may determine this graphically or by calculating derivatives. ii) Determine the bounds (above and below of the sequence). iii) Can you apply the bounded & monotonic theorem for convergence to this sequence? iv) Does this sequence converge by another theorem? If so, which one? v) If the sequence converges, what does it converge to? (20 points)

$$b_n = \frac{\cos n}{n}$$

Cannot apply bounded & monotonic theorem

Sequence is not monotonic
it is bounded above by 1
and below by -1
Can use the squeeze theorem

$-1 \leq \cos n \leq 1$

$\lim_{n\to\infty}\left[\frac{-1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}\right]$

$= 0 \leq \frac{\cos n}{n} \leq 0$ Therefore $\to 0$

4. Use a power series to approximate the integral $\frac{1}{\sqrt{2\pi}}\int_0^1 e^{-x^2/2}\,dx$. Use 6 terms, given that

$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$. Round your answer to 4 decimal places. (10 points)

$$e^{-x^2} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{n!\cdot 2^n}$$

$$\frac{1}{\sqrt{2\pi}}\cdot\int_0^1 1-\frac{x^2}{2}+\frac{x^4}{4}-\frac{x^6}{6\cdot 8}+\frac{x^8}{24\cdot 16}-\frac{x^{10}}{120\cdot 32}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[x-\frac{x^3}{6}+\frac{x^5}{20}-\frac{x^7}{7\cdot 6\cdot 8}+\frac{x^9}{24\cdot 16\cdot 9}-\frac{x^{11}}{11\cdot 120\cdot 32}\right]_0^1 \approx$$

$$\frac{1}{\sqrt{2\pi}}[0.880023] = 0.351318$$

5. What is the maximum error R_n for the Taylor polynomial $-\ln(1-x) \approx x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}$ on the interval $\left[-\frac{1}{2},\frac{1}{2}\right]$. (9 points)

$n=0 \quad 1 \quad 2 \quad 3$

$$R_n = \frac{x^5}{5} = \frac{(1/2)^5}{5} = \frac{1}{160}$$

$-\ln(1-x) = f$

$(1-x)^{-1} = \frac{1}{1-x} = f'$

$(1-x)^{-2} = f''$

$2(1-x)^{-3} = f'''$

$6(1-x)^{-4} = f^{IV}$

$24(1-x)^{-5} = f^{V}$

$24(1-\frac{1}{2})^{-5} = 24(2)^5$

$24(1+1/2)^{-5}$ smaller den.

$$\frac{24(2^5)}{120} = \frac{32}{5} = 6.4$$

Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

6. For each of the series below (same as in #1), state the name of the test used to determine convergence. Show the work here to support your conclusion above. (8 points each)

a. $\sum_{n=1}^{\infty}\ln\left(\frac{n+1}{n}\right)$ telescoping series

$\ln(n+1)-\ln n$ — diverges

$$\ln(2) - \lim_{n\to\infty}\ln(n) = -\infty$$

work in part 1

b. $\sum_{n=1}^{\infty} \frac{n+1}{2n-1}$ nth term test diverges

$\lim_{n \to \infty} \frac{n+1}{2n-1} = \frac{1}{2} \neq 0$

c. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt[3]{(\ln n)^2}}$ integral test $\int_2^{\infty} \frac{1}{n\sqrt[3]{\ln n}^2}\,dn = \infty$

diverges

d. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{3n^2+2}}$ limit comparison w/ $\frac{1}{n}$ diverges by p-series

diverges

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e. $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$ integral test diverges

limit comparison to $\frac{1}{n}$ (diverges, p-series

$\lim_{n \to \infty} \frac{\tan(\frac{1}{n})}{(\frac{1}{n})} = \lim_{x \to 0} \frac{\tan x}{x} = 1$

let $n = \frac{1}{x} \to x = \frac{1}{n}$

f. $\sum_{n=2}^{\infty} \frac{n}{(n^2-1)^2}$ limit comparison w/ $\frac{1}{n^3}$ converges by p-series

converges

g. $\sum_{n=0}^{\infty} \frac{\cos n\pi}{n+1}$ alternating series test

Converges

h. $\sum_{n=1}^{\infty} \frac{(n+1)(-3)^n}{n!}$ ratio test

Converges

i. $\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}$ ratio test converges

j. $\sum_{n=1}^{\infty} \frac{11}{n^{9/8}}$ p-series test

converges

1! 3! 5!

7. For the sequence $1, 6x, 120x^2, 5040x^3, 362880x^4, \ldots.$, find a formula for the nth term of the sequence (starting at n=0). (10 points)

$(2n+1)!\,x^n$

8. Find the interval of convergence of the power series. (10 points each)

$$\sum_{n=1}^{\infty} \frac{(-1)^n n!(x+1)^n}{(2n+1)^2}$$

$\lim_{n\to\infty} \left| \frac{(n+1)!\,(x+1)^{n+1}}{(2n+3)^2} \cdot \frac{(2n+1)^2}{n!\,(x+1)^n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)(x+1)\cdot(2n+1)^2}{(2n+3)^2} \right|$

$= \infty$ unless $x = -1$

$\lim_{n\to\infty} \left| \frac{(2n+1)^2}{(2n+3)^2} \right| = 1$

$\{-1\}$

$\lim_{n\to\infty} (n+1)(x+1) = \infty$ unless $x+1=0$

9. Find the Taylor Polynomial for the function at the indicated value of c. Use the tables provided. (15 points)

$f(x) = \frac{1}{x}, n = 4, c = 1$

n	n!	$f^{(n)}(x)$	$f^{(n)}(c)$	$(x-c)^n$	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	x^{-1}	1	1	$\frac{1}{1}(1)(1) = 1$
1	1	$-x^{-2}$	-1	$(x-1)$	$\frac{-1}{1}(x-1) = -(x-1)$
2	2	$2x^{-3}$	2	$(x-1)^2$	$\frac{2}{2}(x-1)^2 = (x-1)^2$
3	6	$-6x^{-4}$	-6	$(x-1)^3$	$\frac{-6}{6}(x-1)^3 = -(x-1)^3$
4	24	$24x^{-5}$	24	$(x-1)^4$	$\frac{24}{24}(x-1)^4 = (x-1)^4$
5	120				
6	720				

$P_n(x) = 1 - (x-1) + (x-1)^2 - (x-1)^3 + (x-1)^4$

10. Find the power series for the functions below. Write your answers with the sum starting at n=0. (12 points each)

a. $f(x) = \ln(x+1)$

$f'(x) = \frac{1}{x+1}$ $a=1$ $r=(-x)$

$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n$ $\frac{1}{1+r} = \sum_{n=0}^{\infty} (-1)^n r^n$

$\int \frac{1}{1-r}\,dr = -\ln(1-r)$ $\int \frac{1\,dr}{1+r} = \ln(1+r)$

$\sum_{n=0}^{\infty} \int (-1)^n r^n = \sum_{n=0} \frac{(-1)^n r^{n+1}}{n+1} + C$

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}$$

b. $r(x) = \frac{x^4}{(1+2x)^3}$

$\frac{a}{1-r} = a(1-r)^{-1} \to a(1-r)^{-2} \to 2a(1-r)^{-3}$

$a\sum_{n=0} x^n \Rightarrow a\sum_{n=1} n x^{n-1} \to a\sum_{n=2} n(n-1)x^{n-2} = a\sum_{n=0} (n+2)(n+1)x^n$

$$\frac{2a}{(1-r)^3} = \sum_{n=0}^{\infty} a(n+2)(n+1) r^n$$

$a = \frac{1}{2}x^4$

$r = (-2x)$

$$\sum_{n=0}^{\infty} \left(\tfrac{1}{2}x^4\right)(n+2)(n+1)(-2x)^n =$$

$$\sum_{n=0}^{\infty} (-2)^n \cdot \tfrac{1}{2}(n+2)(n+1)x^{n+4}$$

$$\sum_{n=0}^{\infty} (-1)^n 2^{n-1}(n+2)(n+1)x^{n+4}$$