10/13/2022

Limit Comparison Tests (5.4)
Root & Ratio Tests (5.6)

Last time we discussed the direct comparison test. Recall that the direct comparison test required inequalities to point in specific directions in order to make the test valid.

We want to know what a series defined by a_n is doing, and we have a similar series b_n that we know from another test what it does.

If b_n converges, then to apply the direct comparison test, we required that $a_n \leq b_n$ for all n. If b_n diverges, then we required that $a_n \geq b_n$ for all n.

The limit comparison is going to not require inequalities to the do the work, so it has more flexibility than the direct comparison. Instead, we are going to use ratios to do the comparison.

The limit comparison: we want to know what a series defined by a_n is doing, and we have another series b_n that is similar and which we know either converges or diverges by another test.

Then $\lim_{n\to\infty} \frac{a_n}{b_n}$: if the limit is a finite and non-zero value, then the two series converge or diverge together.
However, if the limit is 0 or ∞ then the series do not compare and you need to choose a different b_n (because the comparison is bad) or a different test.

Consider the series

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$$

We can't use the direct comparison here because $\frac{1}{n^2-1} \geq \frac{1}{n^2}$, and $\frac{1}{n^2}$ has a convergent series. This doesn't satisfy the inequality that we needed for the direct comparison.

Instead we use the limit comparison.

$$\lim_{n\to\infty} \frac{\frac{1}{n^2-1}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2-1} = \lim_{n\to\infty} \left[1 + \frac{1}{n^2-1}\right] = 1 + 0 = 1$$

Therefore, since $\sum \frac{1}{n^2}$ converges by the p-series, then the series $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ also converges by the limit comparison test.

It doesn't really matter which way you divide. If the series don't compare, then you will get either 0 or ∞, but if you flip the order of division then you'll just get ∞ or 0 for the same comparison.

Any finite (non-zero) value will still produce a finite, non-zero value in the reciprocal.

How do you pick the series to do the comparison to?

For a rational expression, use the leading terms in the numerator and the denominator. Reduce whatever is left.
If

$$a_n = \frac{(n^2 + n - 1)}{n^4 + 5n^3 - 6}$$

Compare to $b_n = \frac{n^2}{n^4} = \frac{1}{n^2}$

Similarly for something similar to a geometric series
If

$$a_n = \frac{(4^n + n^3)}{7^n - 1}$$

Compare to $b_n = \frac{4^n}{7^n} = \left(\frac{4}{7}\right)^n$

If you have something like

$$a_n = \frac{n - 1}{2^n + n^2}$$

You could compare to $\frac{1}{2^n} = b_n$ or worst case compare to $b_n = \frac{n}{2^n} = n2^{-n}$

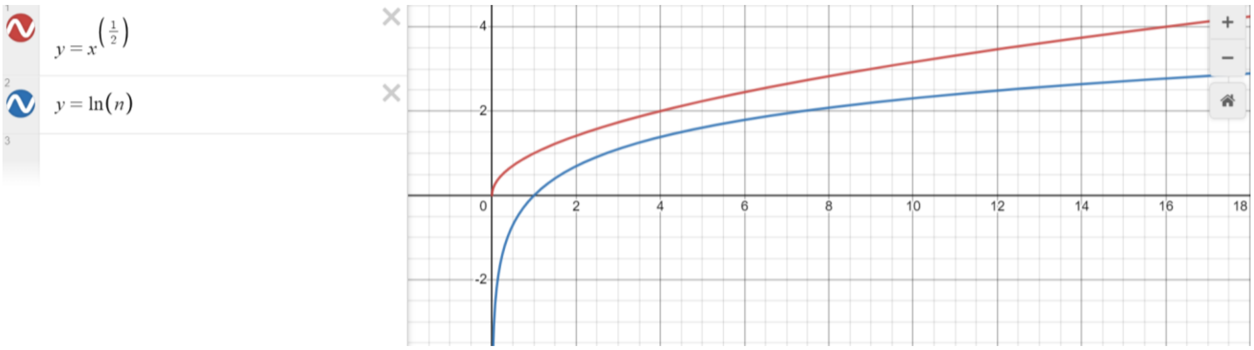
Example.

$$\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$$

We could try to replace $\ln(n)$ with n. That would reduce to $\frac{n}{n^2} = \frac{1}{n}$ diverges. For the direct comparison this wouldn't work because $\ln(n) \leq n$.

$$\lim_{n\to\infty} \frac{\frac{\ln(n)}{n^2}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\ln(n)}{n^2} \times \frac{n}{1} = \lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

Doesn't work because 0 means that the series don't behave similarly.



$$\sqrt{n} \geq \ln(n)$$

$$\frac{\ln(n)}{n^2} \leq \frac{\sqrt{n}}{n^2} = \frac{1}{n^{\frac{3}{2}}}$$

We can do a direct comparison here. Since $\frac{1}{n^{\frac{3}{2}}}$ converges by the p-series, and $\frac{\ln(n)}{n^2} \leq \frac{1}{n^{\frac{3}{2}}}$.

If we did this same test with the limit comparison

$$\lim_{n\to\infty} \frac{\left(\frac{\ln(n)}{n^2}\right)}{\frac{1}{n^{\frac{3}{2}}}} = \lim_{n\to\infty} \frac{\ln(n)}{n^2} \times n^{\frac{3}{2}} = \lim_{n\to\infty} \frac{\ln(n)}{n^{1/2}} = \lim_{n\to\infty} \frac{1/\mathrm{n}}{\frac{1}{2}n^{-\frac{1}{2}}} = \lim_{n\to\infty} \frac{1}{n} \times 2n^{\frac{1}{2}} = \lim_{n\to\infty} 2n^{-\frac{1}{2}} = 0$$

Direct comparison works better with ln(n) than the limit comparison. (or you can try to use maybe an integral test). The limit comparison works better with rational expressions (numerator and denominator are polynomials), or expressions that can “simplify” to geometric series.

Ratio and Root tests (5.6)

Ratio test:
If we have a series defined by a_n. We are going to try to treat the series like a geometric series. We want to find the ratio of consecutive terms and see how it behaves in the limit.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

If the limit is < 1, the series converges. If the limit is > 1 the series diverges. If the limit is equal to 1, the convergence/diverges is inconclusive. (This means you would need another test.)

Root test:
If we have a series defined by a_n. We are going to get the ratio of terms in the series in the limit in a slightly different way.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|}$$

If the limit is < 1, the series converges. If the limit is > 1 the series diverges. If the limit is equal to 1, the convergence/diverges is inconclusive. (This means you would need another test.)

Factorials work best in the ratio test. If you have an expression where most terms are raised to the nth power, then the root test works well (with much less algebra).

Polynomials/rational expressions will be 1 in the limit in both cases. $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

$$\frac{n!}{(n+1)!} = \frac{n!}{n!\,(n+1)} = \frac{1}{n+1}$$

$$\frac{(2n+1)!}{(2n+3)!} = \frac{(2n+1)!}{(2n+1)!\,(2n+2)(2n+3)} = \frac{1}{(2n+2)(2n+3)}$$

Examples.

$$\sum_{n=0}^{\infty} \frac{2^n}{n!}$$

Use the ratio test.

$$\lim_{n\to\infty} \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \lim_{n\to\infty} \frac{2^{n+1}}{(n+1)!} \times \frac{n!}{2^n} = \lim_{n\to\infty} \frac{2^n(2)}{(n+1)n!} \times \frac{n!}{2^n} = \lim_{n\to\infty} \frac{(2)}{(n+1)} = 0$$

This limit is <1, therefore the series converges.

Example.

$$\sum_{n=0}^{\infty} \frac{n^n}{n!}$$

$$\lim_{n\to\infty} \frac{\frac{(n+1)^{n+1}}{(n+1)!}}{\frac{n^n}{n!}} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{(n+1)!} \times \frac{n!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^n(n+1)}{(n+1)n!} \times \frac{n!}{n^n} =$$

$$\lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n$$

$$\ln\left(\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n\right) = \ln(L)$$

$$\lim_{n\to\infty} ln\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{n^{-1}} \lim_{n\to\infty} \frac{\frac{1}{1+\frac{1}{n}} \times \left(-\frac{1}{n^2}\right)}{-\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n}} = 1$$

$$1 = \ln(L)$$
$$L = e$$
$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Since e>1, this series diverges.

Example.

$$\sum_{n=0}^{\infty}\frac{3^{n^2}}{2^{n^3}}$$

Use the root test.

$$\lim_{n\to\infty}\sqrt[n]{\frac{3^{n^2}}{2^{n^3}}} = \lim_{n\to\infty}\frac{3^{\frac{n^2}{n}}}{2^{\frac{n^3}{n}}} = \lim_{n\to\infty}\frac{3^n}{2^{n^2}} = \lim_{n\to\infty}\left(\frac{3}{2^n}\right)^n$$

$$ln\lim_{n\to\infty}\left(\frac{3}{2^n}\right)^n = \ln(L)$$

$$\lim_{n\to\infty} ln\left(\frac{3}{2^n}\right)^n = \lim_{n\to\infty} n\ln 3(2^{-n}) = \lim_{n\to\infty}\ln\frac{3(2^{-n})}{n^{-1}} = \lim_{n\to\infty}\frac{3(-)(2^{-n})\ln 2}{-n^{-2}} = \lim_{n\to\infty}\frac{n^2(3\ln 2)}{2^n} = 0$$

This is less than 1 and so converges.

A more typical root test is $a_n = \left(\frac{2n+1}{3n-2}\right)^{4n}$

So for next time, we will look at ratio test especially used in power series (infinite series that contain x). Then we'll also talk about examples and how to choose a good test. Which ones work, and which ones are the easiest to try first.