10/6/2022

Series Tests Continued

Alternating Series Tests (5.5)
Error estimates (5.3/5.5)
Comparison tests? (5.4)

Alternating Series test
An alternating series looks like:

$$a_n = (-1)^n b_n$$

(b_n is always positive)
Recall that $(-1)^n = \cos(n\pi)$

$\sin(n)$ is not an alternating series because even though the sign is changing, it is not changing with every term.

Alternating Test says that if a_n is an alternating series, then if $\lim_{n\to\infty} a_n = 0$, then the series converges. (If the limit does not go to zero, then the series diverges.)

Compare to the nth-term test or test for divergence:
If a series defined in terms of a_n has $\lim_{n\to\infty} a_n \neq 0$, then the series diverges. If the limit is zero, the test is inconclusive.

The combination here:
If the limit does not go to zero, any series diverges.
If the series is alternating, then if the limit zero the series converges.
Otherwise, inconclusive.

Absolute convergence vs. conditional convergence

Conditional convergence means that the alternating series with converge because it is alternating.
Absolute convergence means that the series will converge even if we remove the alternating component (we take the absolute value).

Conditional convergence: a_n is alternating and $\lim_{n\to\infty} a_n = 0$
Absolute convergence: a_n converges and $|a_n|$ converges.

Alternating harmonic series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$$

The alternating harmonic series converges.
But as we saw in the last class with the integral test, this series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.
The alternating harmonic series is conditionally convergent.

On the other hand:

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\lim_{n\to\infty}\frac{(-1)^n}{n^2}=0$$

Does converge by the alternating series test.
If we take the absolute value:

$$\sum_{n=1}^{\infty}\frac{1}{n^2}$$

And by the p-series, this series converges also.
So our original series converges absolutely.

The limit going to zero does not tell us anything after taking the absolute value. We need to apply a different test.

We can't apply the integral test if the series is alternating.
$(-1)^x$ is not a continuous function, and therefore we can't integrate it. If we want to apply the integral test, we have to first apply the absolute value.

Error calculation for the alternating series test:

$$E \leq |a_{n+1}|$$

If you stop counting at n terms, the maximum possible error is the absolute value of the next term.

For the integral test error:

$$E \leq \int_{N}^{\infty} f(x)dx$$

With $a_n = f(n)$ and N being the number of terms you stopped counting at.
Maximum error is the tail of the integral after you stop counting.

Find the error on the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

For 15 terms.

$$E \approx \left|\frac{(-1)^{16}}{16^2}\right| = 0.0039 \ldots \approx 0.004$$

If I apply the integral test to this problem it would be incorrect because I can't integrate with the $(-1)^n$.

If I wanted the error on

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Then I would use the integral test.
Estimate the error after 15 terms.

$$E \approx \int_{15}^{\infty} \frac{1}{x^2} dx = -\frac{1}{x}\bigg|_{15}^{\infty} = 0 + \frac{1}{15} = 0.06666 \ldots$$

To determine the number of terms needed to ensure that the sum was within a given error range.

Estimate the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

To within $E \leq 0.00001 = 10^{-5}$

$$\left|\frac{(-1)^n}{n^2}\right| \approx 10^{-5}$$

$$\frac{1}{n^2} \approx 10^{-5}$$

$$n^2 \approx 10^5$$

$$n \approx \sqrt{10^5} \approx 316.227766 \ldots$$

$$n = 317$$

Estimate the number of terms needed to find the sum of

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

To withing $E \leq 10^{-5}$. I would need to use the integral test. After integrating:

$$10^{-5} \approx 0 + \frac{1}{N}$$

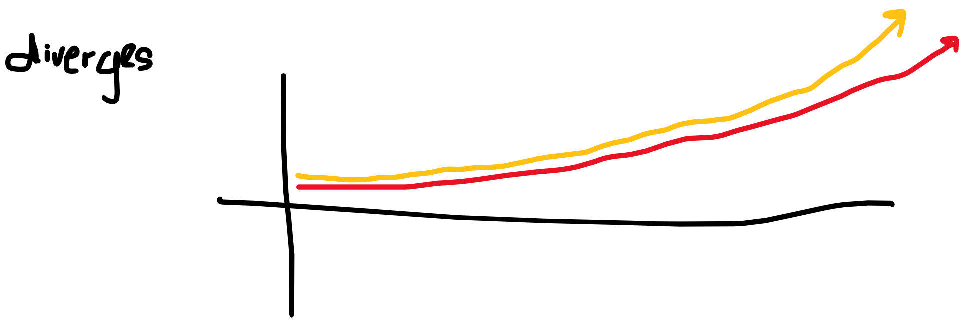
$$N = 10^5 = 100{,}000$$

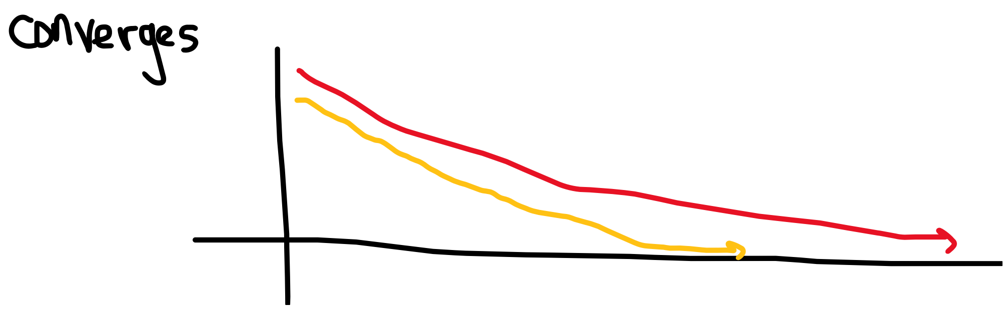
I would need 10,000 terms to find the value of the sum to this level of accuracy.

Comparison Tests (5.4)

Two comparison tests:
Direct comparison test and the limit comparison

Direct comparison test says that if I have two series, a_n and b_n are both positive, and $a_n \leq b_n$ and b_n converges, then a_n also converges. If $a_n \geq b_n$ and b_n diverges, then a_n also diverges.





Inequalities are very strict in the direct comparison test.
For convergence, k as a constant

$$\frac{1}{n} \; vs. \frac{1}{n+k}$$

Which of these is bigger?

$$\frac{1}{n-1} \geq \frac{1}{n}$$

If you are proving divergence, then subtraction in the denominator makes the expression larger, use a simplified version of the same expression for the direct comparison

$$\frac{1}{n^2} \geq \frac{1}{(n+1)^2}$$

Or

$$\frac{1}{n^2} \geq \frac{1}{n^2+1}$$

If you are proving convergence, then addition in the denominator makes the expression smaller, use a simplified version of the same expression for the direct comparison

In the numerator subtraction makes something smaller, and addition makes it bigger.

$$\frac{n+1}{n^3} \geq \frac{1}{n^2}$$

This would not work as a direct comparison, because you want the convergent series that you know to be bigger than the series you are comparing it to.

Example.

$$\sum_{n=1}^{\infty} \frac{1}{2^n + 1}$$

Consider that $\sum_{n=1}^{\infty} \frac{1}{2^n}$ this is a geometric series, r=1/2. Therefore it converges. This is very similar, but not identical. Can we show that the series of our example are smaller than the geometric series? If so, we can use the direct comparison to show convergence.

$$\frac{1}{2^n} \geq \frac{1}{2^n + 1}$$

???
n=1, $\frac{1}{2} \geq \frac{1}{3}$
n=2, $\frac{1}{4} \geq \frac{1}{5}$
etc.

So, by the direction comparison, the $\frac{1}{2^n+1} \leq \frac{1}{2^n}$, and since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges by the geometric series test, we also know that the series $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ converges also.

Example.

$$\sum_{n=3}^{\infty} \frac{\sqrt{n}+1}{n-1}$$

Think about possibly comparing to

$$\sum_{n=3}^{\infty} \frac{\sqrt{n}}{n} = \sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$$

Is this true?

$$\frac{\sqrt{n}+1}{n-1} \geq \frac{1}{\sqrt{n}} = \frac{\sqrt{n}}{n}$$

$$\frac{\sqrt{n}+1}{n-1} \geq \frac{\sqrt{n}+1}{n} \geq \frac{\sqrt{n}}{n}$$

This is true. The numerator is bigger on the left, so the whole fraction is bigger. If I make the denominator smaller, the whole fraction is bigger.

The series $\frac{\sqrt{n}+1}{n-1} \geq \frac{1}{\sqrt{n}}$, and therefore, since $\sum_{n=3}^{\infty} \frac{1}{\sqrt{n}}$ diverges by the p-series test, the series $\sum_{n=3}^{\infty} \frac{\sqrt{n}+1}{n-1}$ also diverges.

The limit comparison is similar in concept, but it doesn't rely on inequalities, it relies on a ratio, and this will make the test less sensitive to the additions and subtractions in our series.

We will pick that up next time.
There is no class on Tuesday for Fall break.