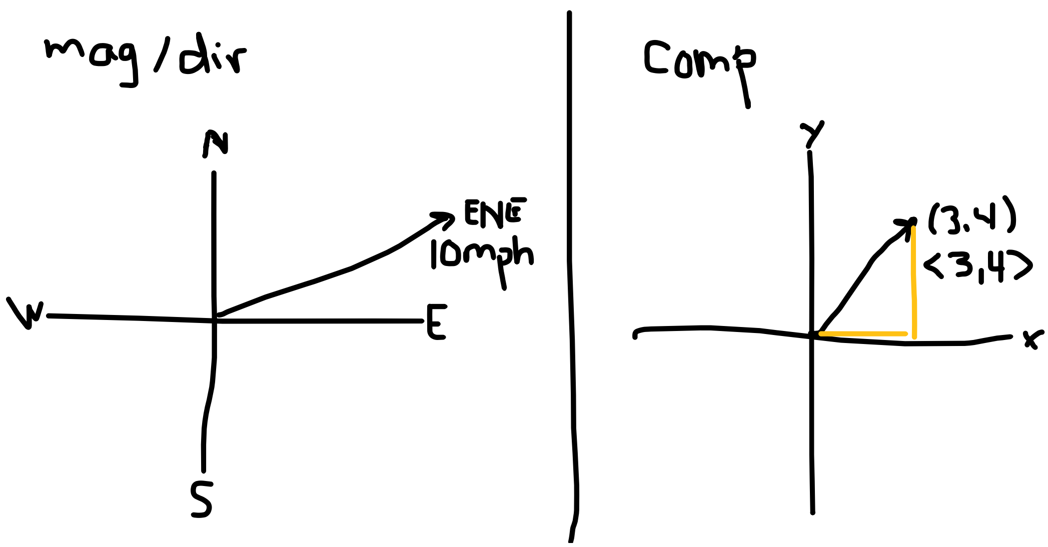
11/15/2022

Parametric Curves (7.1)
Vectors
Calculus of Parametric Functions (7.2) ???

Vectors
An object that has both a magnitude and a direction. And it obeys a set of properties (linear algebra). In 2D, the magnitude can be thought as the "radius" and the direction as the angle. Another way to express a vector is in component form. In certain fixed direction, what is the magnitude.

Wind direction/speed is an example of the magnitude/direction form.
A point (a the line connecting it to the origin) is a component form.



Suppose we have a vector $\vec{v}$ or $\boldsymbol{v}$. The magnitude in some formulas will be written as $r = \|\vec{v}\|$. The direction but typically is given as θ.
Once we get out of 2D, then it becomes more difficult to express the direction. Sometimes the direction will be given in component form as "unit vector".

The component form of the vector $\vec{v} =< x, y >= \langle 3,4 \rangle = 3\hat{\imath} + 4\hat{\jmath}$

$$\vec{v} = \langle 1, -6 \rangle$$

What is the magnitude of this vector?

$$\|\vec{v}\| = \sqrt{a^2 + b^2}$$

$$\|\vec{v}\| = \sqrt{1^2 + (-6)^2} = \sqrt{37}$$

What is the direction of this vector?

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

$$\theta = \tan^{-1}\left(-\frac{6}{1}\right) \approx -1.4056 \ldots$$

If the vector is in the 2^{nd} or 3^{rd} quadrant, add π to the value (or 180 if in degrees).

To go from magnitude and direction form to the component form follow the formula:

$$\vec{v} = \langle r\, cos\theta, r \sin\theta \rangle = r\cos\theta\, \hat{\imath} + r\sin\theta\, \hat{\jmath}$$

Suppose I have a force of 100 N being applied at angle of 30-degrees to the positive x-axis. What is the component form of the vector?

$$\vec{F} = \langle 100\cos 30°, 100\sin 30° \rangle = \langle \frac{100\sqrt{3}}{2}, 100\left(\frac{1}{2}\right) \rangle = \langle 50\sqrt{3}, 50 \rangle$$

In order to add vectors together, we need to have them in component form. Vectors add component by component.

$$\langle 1,2 \rangle + \langle 3,-5 \rangle = \langle 4,-3 \rangle$$

If I want to add to force vectors together, I need to convert them to component form, add them, and then convert them back to magnitude and direction form to get the sum of the forces.

Unit vector.
Is a vector that has length = 1. We can obtain it from any vector by dividing the vector by it's length.

$$\hat{v} = \frac{\vec{v}}{||\vec{v}||}$$

$$\vec{v} = \langle 1, -6 \rangle$$

$$\hat{v} = \langle \frac{1}{\sqrt{37}}, -\frac{6}{\sqrt{37}} \rangle$$

This is a unit vector in the direction of the vector $\vec{v}$.

Dot product/inner product/scalar product

$$\langle a,b \rangle \cdot \langle c,d \rangle = \vec{v} \cdot \vec{w} = ac + bd$$

$$\langle 1,-6 \rangle \cdot \langle 2,3 \rangle = 2 - 18 = -16$$

The value of the dot product is related to the angle between the vectors.

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{||\vec{u}||\,||\vec{v}||}$$

$$||\vec{v}|| = \sqrt{2^2 + 3^2} = \sqrt{13}$$

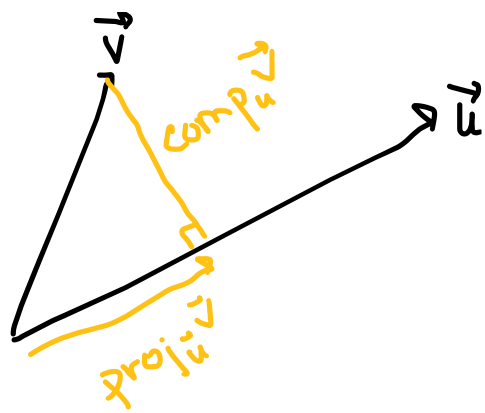
$$\cos\theta = -\frac{16}{\sqrt{37}\sqrt{13}} \approx -.7295\ldots$$

$$\theta \approx 136.85°, 2.388\ radians$$

The angle between the vectors is obtuse.
When the dot product is negative, the angle between the vectors is obtuse.
When the dot product is positive, the angle between the vectors is acute.
When the dot product is zero, the angle between the vector is 90-degrees, or a right angle.

Projection
If I want to project the vector $\vec{v}$ onto the vector $\vec{u}$, use the projection formula:

$$proj_{\vec{u}}(\vec{v}) = \left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|^2}\right)\vec{u}$$



$$\vec{u} = \langle 1, -6\rangle, \vec{v} = \langle 2,3\rangle$$

$$proj_{\vec{u}}(\vec{v}) = \left(-\frac{16}{\left(\sqrt{37}\right)^2}\right)\langle 1, -6\rangle = -\frac{16}{37}\langle 1, -6\rangle = \langle -\frac{16}{37}, \frac{96}{37}\rangle$$

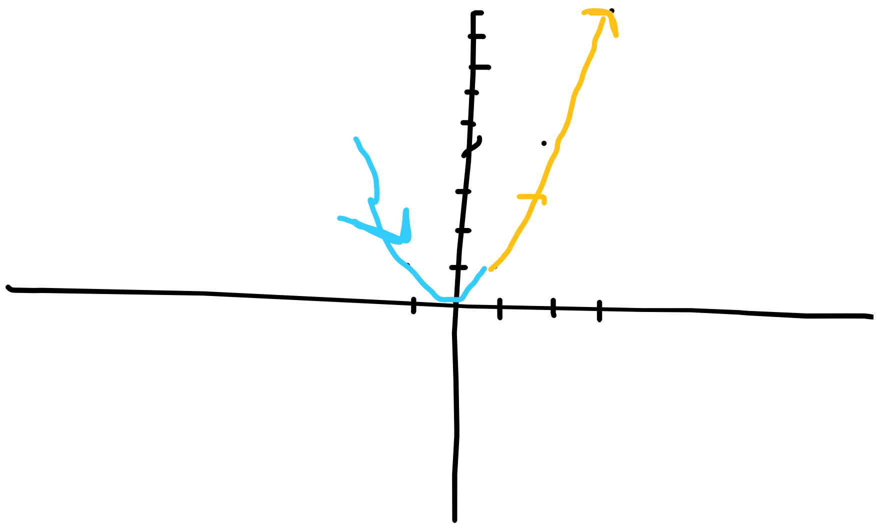
$$comp_{\vec{u}}(\vec{v}) = \vec{v} - proj_{\vec{u}}(\vec{v}) = \langle 2,3\rangle - \langle -\frac{16}{37}, \frac{96}{37}\rangle = \langle \frac{90}{37}, \frac{15}{37}\rangle$$

Work can be calculated by $W = \vec{F}\cdot\vec{d}$

Parametric equations.
A set of equations where x and y are both functions of a third variable (parameter), but are not necessarily a function in x and y alone.

$$x(t) = t + 1$$
$$y(t) = (t+1)^2$$

To plot the curve, plug in values for t and obtain a pair of points in x and y. Plot the points. Connect the dots. The curve has an orientation: is the direction in which t is increasing.



https://www.geogebra.org/m/cAsHbXEU

Standard parameterizations.

Suppose you have a function $y(x)$ and you want to create a parametric version of that function. Typically, you let $x(t) = t$, and then $y(t) = y\big(x(t)\big)$.

Suppose I want to parametrize $y = x^3 - 6x$. Let $x = t$, then $y = t^3 - 6t$. And that's it!

We can use vectors as a "wrapper" for the set of equations.

$$\vec{r}(t) = \langle x(t), y(t) \rangle = \langle t, t^3 - 6t \rangle$$

Can do vector operations on this expression. I can add it to another function in the same form. I can find the distance to the origin for any point in space (magnitude of the vector).

What if I want to parametrize a straight line, but I don't want to find $y = mx + b$?

$$(1,3), (4,7)$$

Find a parametric equation of the line that connects the two points.

$$x = at + x_0$$
$$y = bt + y_0$$
$$\langle a, b \rangle = \langle \Delta x, \Delta y \rangle$$

$$a = 4 - 1 = 3, x_0 = 1$$
$$b = 7 - 3 = 4, y_0 = 3$$

$$\vec{r}(t) = \langle 3t + 1, 4t + 3 \rangle$$

If I want to go back to function form, I need to solve one equation (of the set) for t, and replace that expression in other function.

$$x = 3t + 1$$
$$x - 1 = 3t$$
$$\frac{x-1}{3} = t$$

$$y = \frac{4(x-1)}{3} + 3$$

This, if simplified, will get me back to the $y = mx + b$ form of the line.

Circles:

$$x = r\cos(t)\,, y = r\sin\,(t)$$

$$x^2 + y^2 = 4$$

Then the parameterized version of this is

$$\langle 2\cos t\,, 2\sin t\rangle$$

Ellipses are similar except the constant radius on each component is replaced by a and b.

$$\langle 3\cos t\,, 4\sin t\rangle$$

Can apply transformations, but signs will match the movement (unlike in (x,y) form).

Matching version for hyperbolas. You can use hyberbolic trig functions or tangent and secant.

Next time we'll do the calculus!!