8/23/2022

Introduction to the Course
Area between Curves (2.1)

Area between two curves:

Method 1 (conceptually):

The formula for integrating (finding the area) between two curves

$$\int_a^b [f(x) - g(x)]\, dx$$

Before, g(x), the second function was always $g(x) = 0$, or $y = 0$.

That simplified our expression to

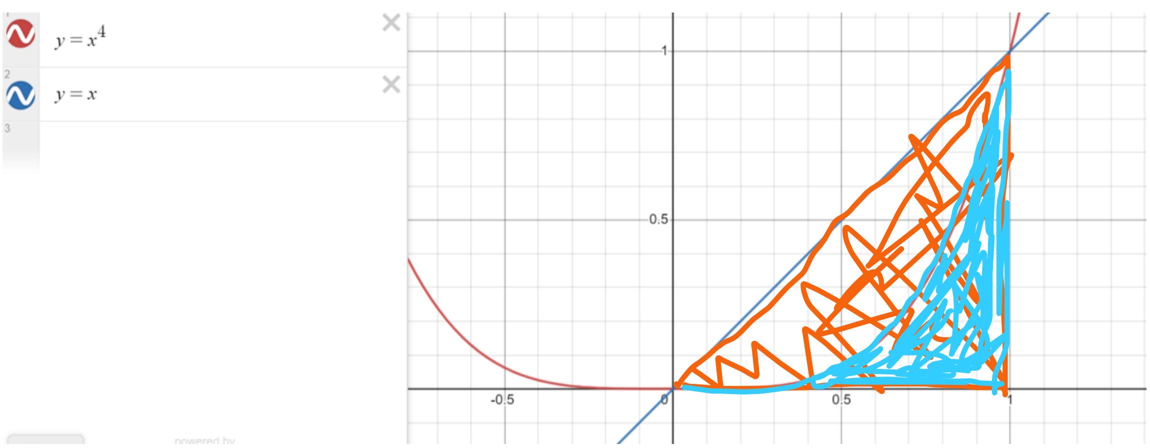
$$\int_a^b [f(x) - g(x)]\, dx = \int_a^b [f(x) - 0]\, dx = \int_a^b [f(x)]\, dx$$

Method 2 (conceptually):

Example:

$$f(x) = x^4, g(x) = x$$

What is the area between f(x) and g(x)?



Find area below top curve, subtract area below the bottom curve = area between the two curves.

If the top curve is f(x), and the bottom curve is g(x), then area under the top curve is $\int_a^b f(x)dx$ and the area under the bottom curve is $\int_a^b g(x)dx$

Then the area between the two curves is

$$\int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$$

In some problems, you will be given endpoints to work with. If you have them, you should use them.

If you subtract in the wrong order, you will get a negative that you should not have. Make sure you drop it for you answer.

If you have no endpoints provided (or only one), then you obtain the endpoints by setting the curves equal to each other and finding the intersections.

Consider our previous example of $f(x) = x, g(x) = x^4$. We want to find the area bounded by the two curves.

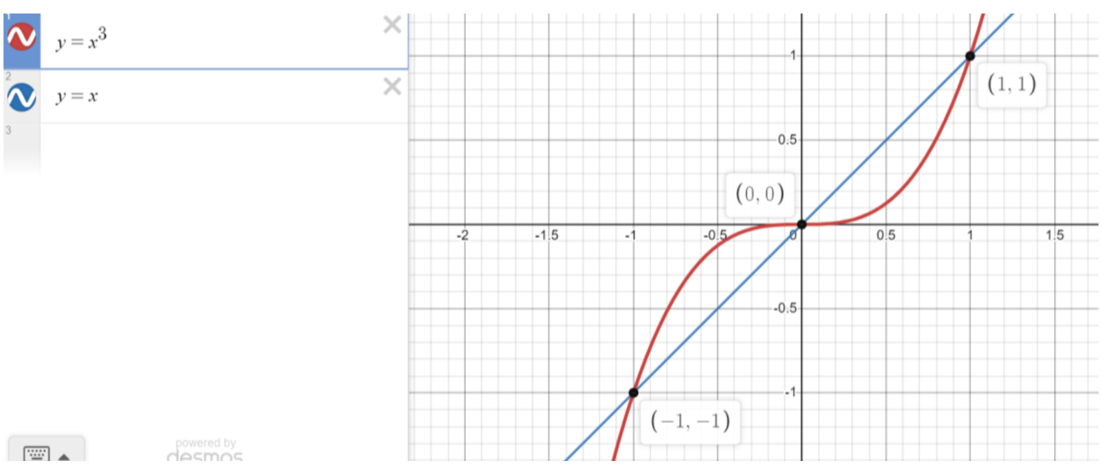
$$x = x^4$$
$$x^4 - x = 0$$
$$x(x^3 - 1) = 0$$

$$x = 0, x = 1$$
$$\int_0^1 x - x^4 dx = \frac{1}{2}x^2 - \frac{1}{5}x^5\Big|_0^1 = \frac{1}{2} - \frac{1}{5} = \frac{3}{10}$$

What if you get 3 intersections?

$$f(x) = x, g(x) = x^3$$

$$x = x^3$$
$$x^3 - x = 0$$
$$x(x^2 - 1) = 0$$
$$x(x-1)(x+1) = 0$$
$$x = 0, -1, 1$$



Bottom region

$$\int_{-1}^{0} x^3 - x dx$$

Top region

$$\int_{0}^{1} x - x^3 dx$$

Because of symmetry

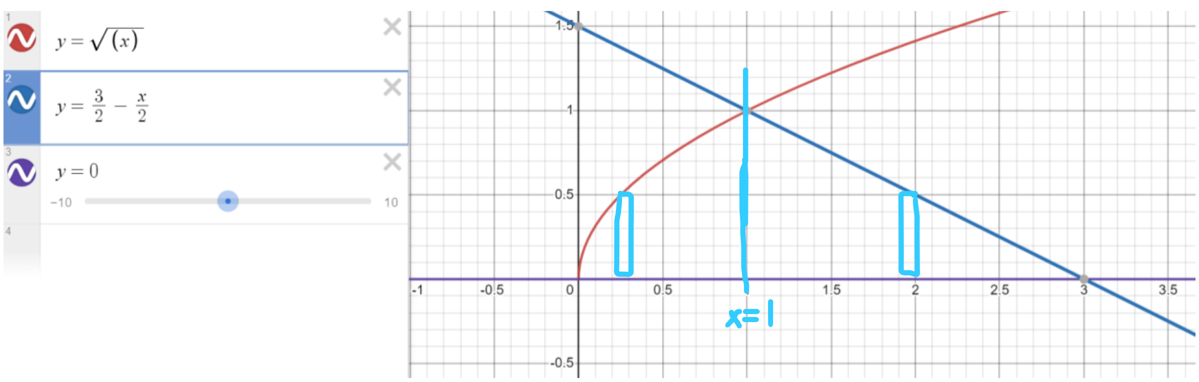
$$\int_{-1}^{0} x^3 - x dx + \int_{0}^{1} x - x^3 dx = 2\int_{0}^{1} x - x^3 dx$$

$$2\int_{0}^{1} x - x^3 dx = 2\left[\frac{1}{2}x^2 - \frac{1}{4}x^4\right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{4}\right] = 2\left[\frac{1}{4}\right] = \frac{1}{2}$$

If you don't split this into two integrals with the order of subtraction flipped, you will get 0, and we can see from the graph that the area between the curves is not 0.

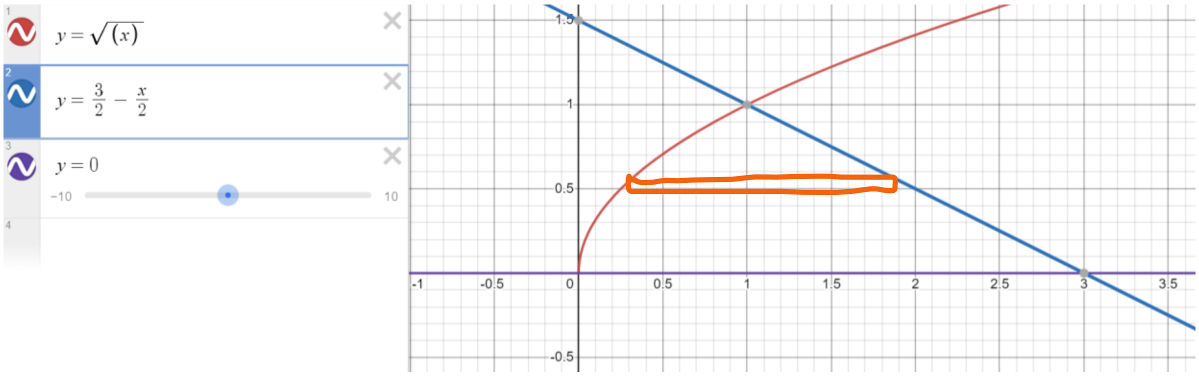
Example.
Find the area bounded by

$$y = \sqrt{x}, y = \frac{3}{2} - \frac{x}{2}, y = 0$$



$$A = \int_0^1 \sqrt{x} dx + \int_1^3 \frac{3}{2} - \frac{1}{2} x dx$$

$$= \int_0^1 x^{\frac{1}{2}} dx + \int_1^3 \frac{3}{2} - \frac{1}{2} x dx = \frac{2}{3} x^{\frac{3}{2}}\Big|_0^1 + \frac{3}{2}x - \frac{1}{4}x^2\Big|_1^3 = \frac{2}{3} + \frac{9}{2} - \frac{9}{4} - \frac{3}{2} + \frac{1}{4} = \frac{5}{3}$$



If we change the orientation, we only need one integral.
Top = rightmost
Bottom = leftmost
We using x as a function of y.

Solve the equations for x, i.e. x(y)

$$y = \sqrt{x} \to x = y^2$$
$$y = \frac{3}{2} - \frac{x}{2} \to 2y = 3 - x \to x = 3 - 2y$$

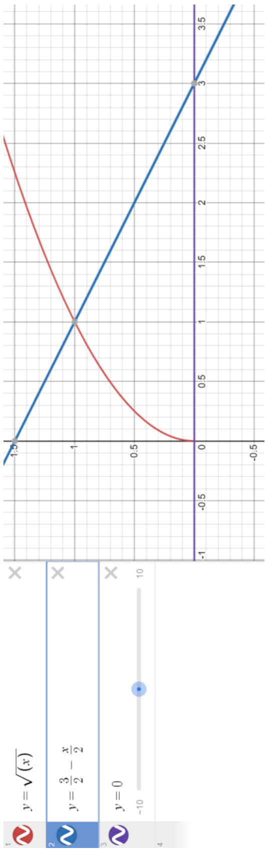
Still have y=0, and I need one more endpoint for the limit = the intersection. Which is at y=1.
Or $y^2 = 3 - 2y$

$$y^2 + 2y - 3 = 0$$
$$(y + 3)(y - 1) = 0$$
$$y = -3, 1$$

$$A = \int_{smallest\ y}^{biggest\ y} right_{function} - left_{function} dy = \int_0^1 3 - 2y - y^2 dy$$

$$= 3y - y^2 - \frac{1}{3}y^3 \Big|_0^1 = 3 - 1 - \frac{1}{3} = \frac{5}{3}$$

If your problems are given to in terms of x as a function of y, then integrate in y most of the time.



Trick 1: rotate your graph.
Trick 2: switch all your variables. You have to do this in your equations and in any limits (constants functions) provided.

$$y = \sqrt{x} \rightarrow x = \sqrt{y}$$
$$y = \frac{3}{2} - \frac{x}{2} \rightarrow x = \frac{3}{2} - \frac{y}{2}$$

In the resolved problems:

$$x = y^2 \rightarrow y = x^2$$
$$x = 3 - 2y \rightarrow y = 3 - 2x$$

But also:

$$y = 0 \rightarrow x = 0$$