8/25/2022

Solids of Revolution (2.2, 2.3)

Two methods: Disk Method (Washer Method) -- method of Slicing
Second method: Cylindrical Shells method, or just Shells method

Suppose I want to rotate the function $y = 4 - 2x$ around the x-axis. Find the volume of revolution bounded by this function, the x-axis and the y-axis.

Volume of one disk:

$$Volume = Area \times height = \pi(radius)^2 \times \Delta x$$

Add up the volumes of our disks to obtain the estimate for the volume

$$\sum_{i=1}^{n} \pi\big(f(x_i)\big)^2 \Delta x$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \pi\big(f(x_i)\big)^2 \Delta x = \int_a^b \pi[f(x)]^2 dx$$

Example. Find the volume of the above cone.

$$V = \pi \int_0^2 (4-2x)^2 dx = \pi \int_0^2 16 - 16x + 4x^2 dx = \pi \left[16x - 8x^2 + \frac{4}{3}x^3\right]_0^2 =$$

$$\pi\left[32 - 32 + \frac{32}{3}\right] = \frac{32}{3}\pi$$

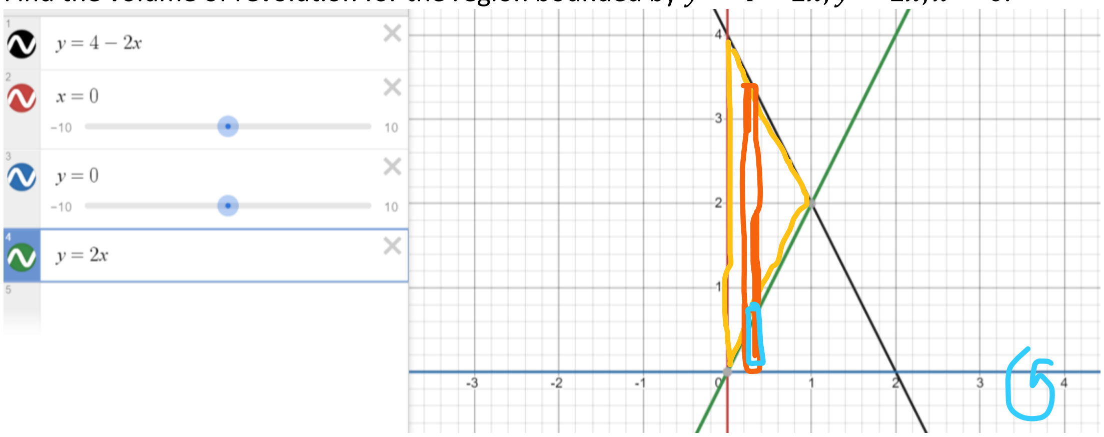
Volume of a cone: $V = \frac{1}{3}\pi r^2 h$

$$\frac{1}{3}\pi(4)^2(2) = \frac{32\pi}{3}$$

Washer Method is an extension of the disk method, where the axis of rotation is not one of the boundaries of the region to be rotated.

$$V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2)dx$$

Example.
Find the volume of revolution for the region bounded by $y = 4 - 2x, y = 2x, x = 0$.



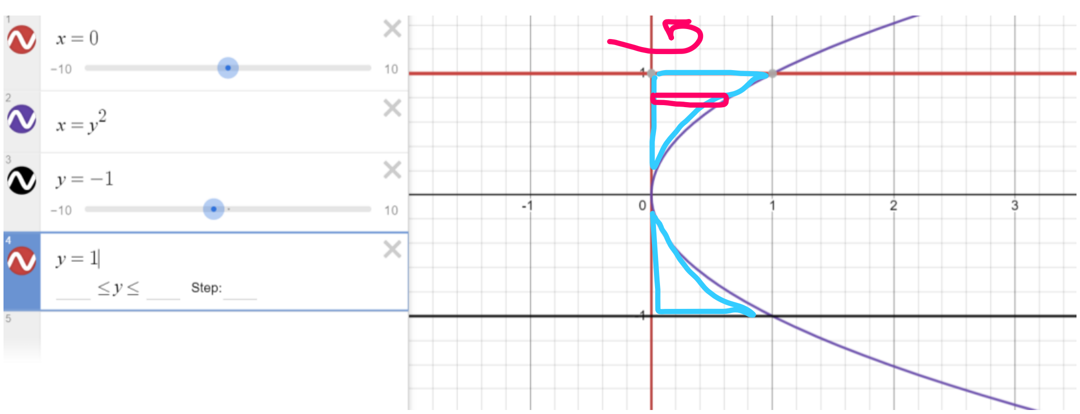
$$V = \pi \int_0^1 (4-2x)^2 dx - \pi \int_0^1 (2x)^2 dx = \pi \int_0^1 (4-2x)^2 - (2x)^2 dx =$$

$$\pi \int_0^1 16 - 16x + 4x^2 - 4x^2 dx = \pi \int_0^1 16 - 16x dx = \pi[16x - 8x^2|_0^1 = \pi(16-8) = 8\pi$$

Disk/Washer Method: Functions of x, rotated around the x-axis.

If you have a function of y, then use the disk/washer method when you are rotating around the y-axis.

Example. The function $x = y^2, -1 \le y \le 1$, and x=0, rotate the region around the y-axis. Find the volume of the solid of revolution.



$$V = (2)\pi \int_0^1 (y^2)^2 dy = 2\pi \int_0^1 y^4 dy = 2\pi \frac{1}{5} y^5 \Big|_0^1 = \frac{2\pi}{5}$$

(twice the top part owing to symmetry).

What if we use an axis of rotation that is not the x- or y-axis?
What axis is the alternate axis of rotation parallel to? That will determine the general form (which variable to use, etc.). We have account for the different distances to the function from the new axis of rotation.

What is the volume of the solid if we rotate the region bounded by $y = 4 - 2x, x = 0, y = 0$ around the line $y = -1$.

Outer radius: the height of the function + the distance to the new axis of rotation.

$$function - axis\_of\_rotation$$
$$4 - 2x - (-1) = 5 - 2x$$

Inner radius: the inner radius (used to be y=0) + the distance to the new axis of rotation

$$0 - (-1) = 1$$

When we rotated around x-axis:

$$V = \pi \int_0^2 (4 - 2x)^2 - 0^2 dx$$

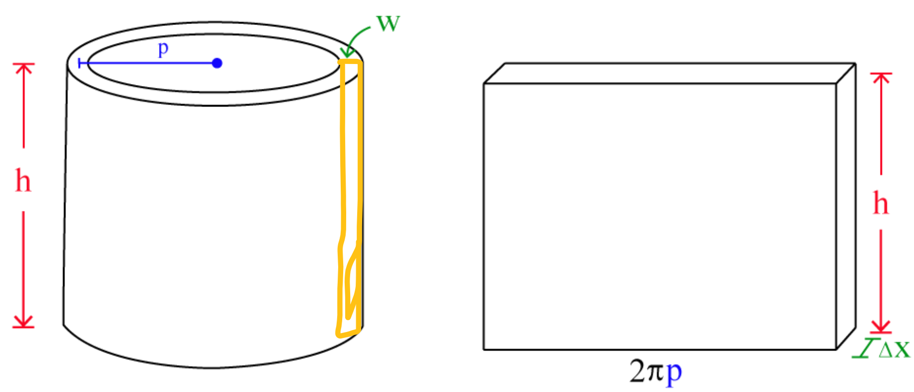
With the new axis of rotation $y = -1$

$$V = \pi \int_0^2 \left(4 - 2x - (-1)\right)^2 - \left(0 - (-1)\right)^2 dx = \pi \int_0^2 (5 - 2x)^2 - (1)^2 dx$$

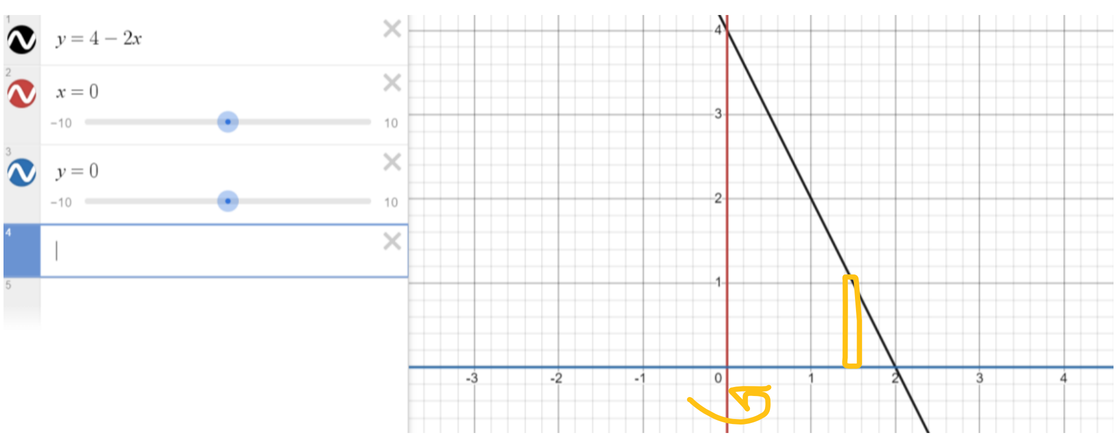
When the axis of rotation is not zero, $y = constant$ is parallel to the x-axis, and so set up your integrals in terms of x.

When the axis of rotation is not zero, $x = constant$ is parallel to the y-axis, and so set up your integrals in terms of y.

Method of Cylindrical Shells (Shells): if you have a function of x and want to rotate around y-axis.
(if you have a function of y and want to rotate around the x-axis, use this method).



Rotate the region bounded by $y = 4 - 2x, x = 0, y = 0$ around the y-axis.

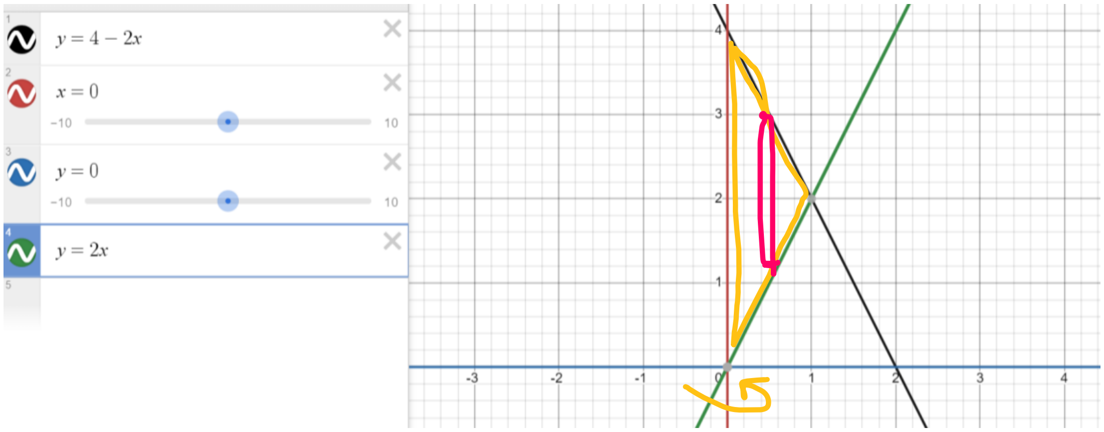


Height of the cylindrical shell is the height of the function $f(x_i)$
What about the width of the shell: this is the width of the rectangle: Δx
P is the radius of the cylinder: the radius of the cylinder is the distance from where the rectangle is to the axis of rotation: x_i

$2\pi r$ is the circumference of the top of the cylinder, and is the length of the rectangular solid.

$$\lim_{n\to\infty}\sum_{i=1}^{n} 2\pi(x_i)f(x_i)\Delta x = 2\pi\int_a^b xf(x)dx$$

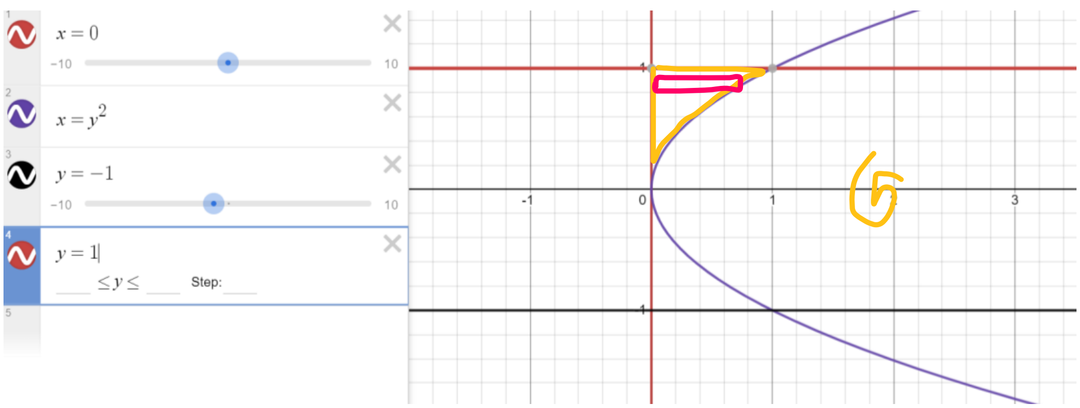
$$V = 2\pi\int_0^2 x(4-2x)dx = 2\pi\int_0^2 4x - 2x^2dx = 2\pi\left[2x^2 - \frac{2}{3}x^3\right]_0^2 = 2\pi\left(8 - \frac{16}{3}\right) = \frac{16\pi}{3}$$



Rotate the region bounded by $y = 4 - 2x, y = 2x, x = 0$ around the y-axis.

$$V = 2\pi \int_0^1 x[(4-2x) - (2x)]dx = 2\pi \int_0^1 x[4-4x]dx = 2\pi \int_0^1 4x - 4x^2 dx$$

If you are rotating around the x-axis, you need your function to be in terms of y.



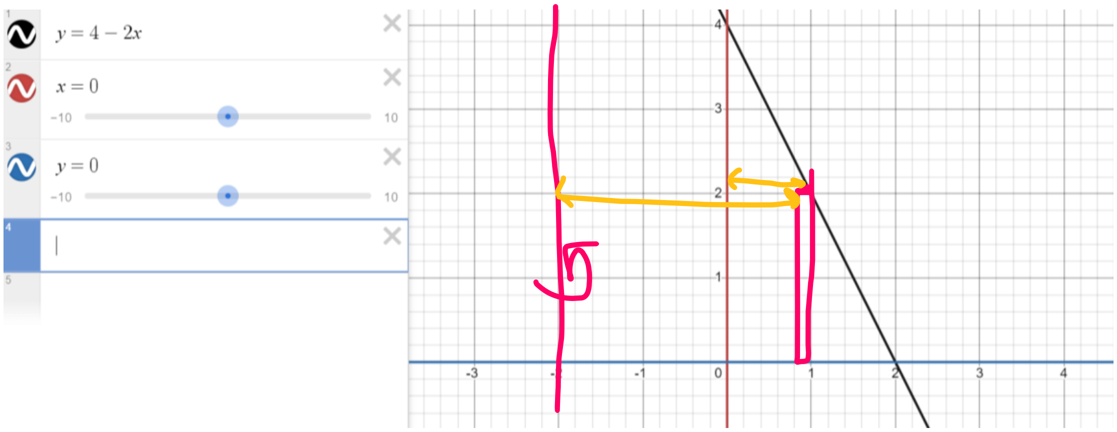
Rotate the region bounded by $x = y^2, y = 1, x = 0$ around the x-axis.

$$V = 2\pi \int_0^1 y(y^2)dy$$

What if we are rotating around a non-zero axis?

It's the radius of the shell that changes, not the height.

Suppose I want to rotate the region bounded by $y = 4 - 2x = 0, y = 0$ around the line $x = -2$?
What if we rotated around $x = 3$?

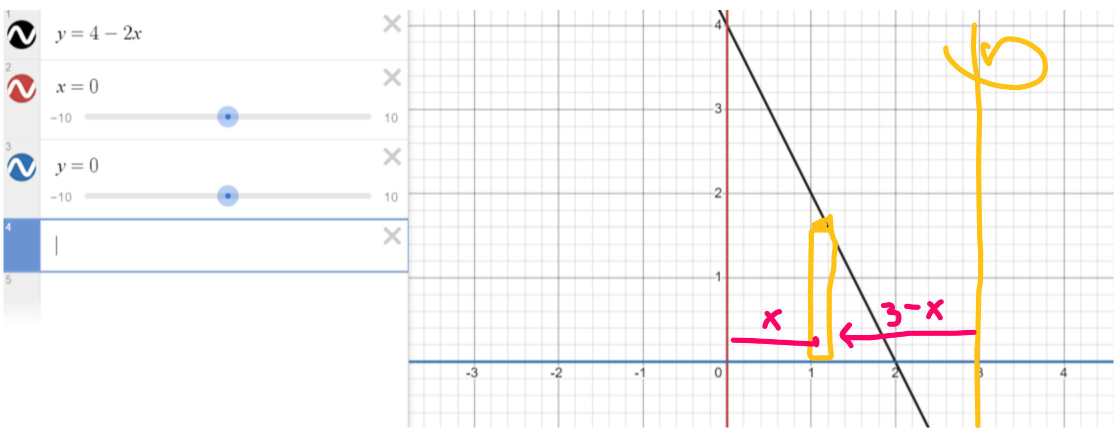


The new “radius” of the shell is old radius (x) minus the axis of rotation.

$$x - (-2) = x + 2$$

Volume:

$$V = 2\pi \int_0^2 (x+2)(4-2x)dx$$



The new radius is the axis of rotation minus x.

If you always do x minus axis of rotation, then you will get the correct answer except for there will be an extra negative. Drop it.

Next time we are going to talk about Arc Length and the surface area of solids of revolution.