9/22/2022

Improper Integrals
Review for Exam #1

Improper Integrals are integrals for which one limit (or both) is infinity, or the function is not defined at one limit, or both, or anywhere inside the integral to be evaluated.

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx$$

Or

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$$

Or

$$\int_{-1}^{1} \frac{1}{\sqrt[3]{x^2}}\,dx$$

Our general idea is to replace any problematic points in the limits with a dummy variable, complete the integration, and then take the limit as our dummy variable approaches the original value.

Example.

$$\int_{1}^{\infty} \frac{1}{x}\,dx$$

$$\lim_{b\to\infty}\int_{1}^{b} \frac{1}{x}\,dx = \lim_{b\to\infty}[\ln x]_1^b = \lim_{b\to\infty}\ln(b) - \ln(1) = \lim_{b\to\infty}\ln(b) = \infty$$

When the value of the integral equals infinity (or remains undefined), then we say that the integral diverges. If the limit can be evaluated to a specific number, then we say that the integral converges.

Example.
Determine if the integral converges or diverges. If it converges, what value does it converge to.
Evaluate the following integral, or determine if it fails to converge.

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}}\,dx$$

$$\lim_{a\to-1}\int_{a}^{0} \frac{1}{\sqrt{1-x^2}}\,dx + \lim_{b\to1}\int_{0}^{b} \frac{1}{\sqrt{1-x^2}}\,dx = \lim_{a\to-1}[\arcsin(x)]_a^0 + \lim_{b\to1}[\arcsin(x)]_0^b =$$

$$\lim_{a\to-1}\arcsin(0) - \arcsin(a) + \lim_{b\to1}\arcsin(b) - \arcsin(0) =$$

$$\lim_{a\to-1} -\arcsin(a) + \lim_{b\to1}\arcsin(b) = -\arcsin(-1) + \arcsin(1) = -\left(-\frac{\pi}{2}\right) + \frac{\pi}{2} = \pi$$

Example.

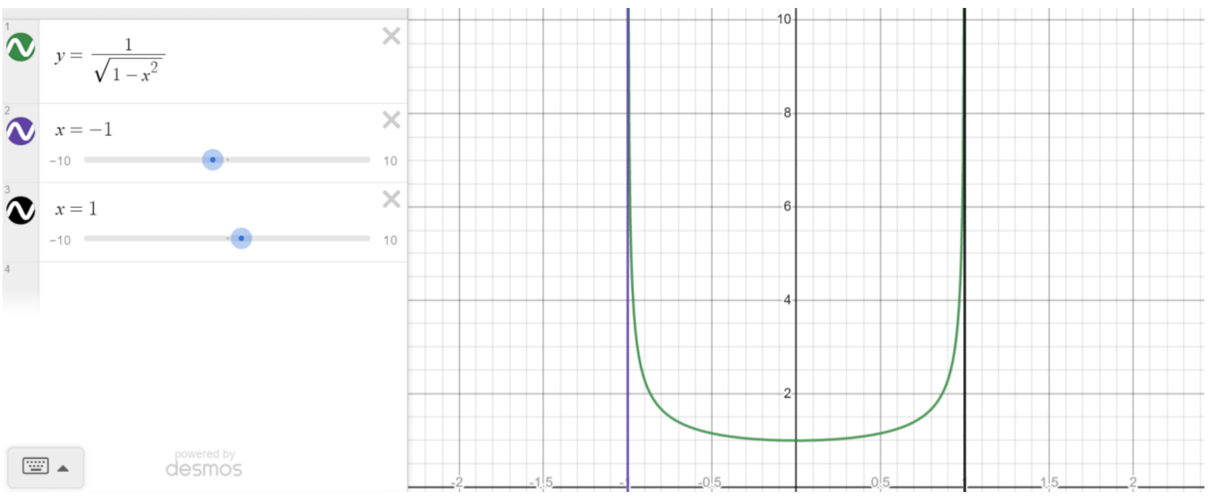
$$\int_{-\infty}^{\infty} xe^{-x^2}dx$$

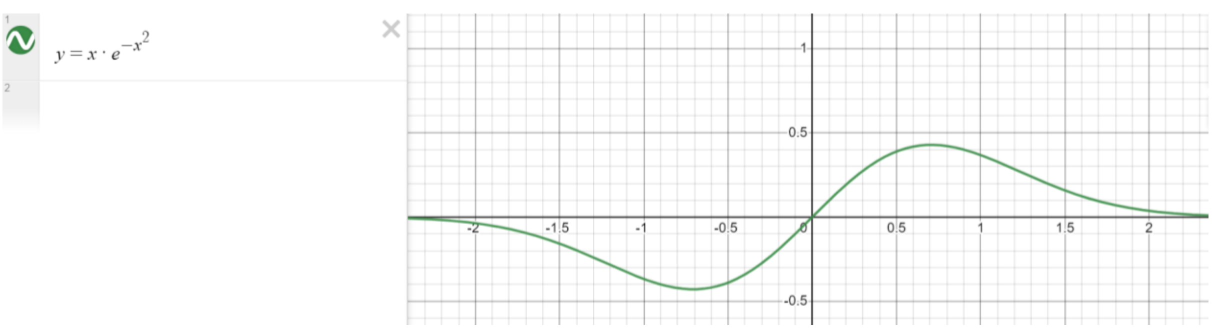
$$\lim_{a\to-\infty}\int_a^0 xe^{-x^2}dx+\lim_{b\to\infty}\int_0^b xe^{-x^2}dx$$

$$u=-x^2, du=-2xdx, -\frac{1}{2}du=xdx$$

$$\lim_{a\to-\infty}-\frac{1}{2}\left[e^{-x^2}\right]_a^0+\lim_{b\to\infty}-\frac{1}{2}\left[e^{-x^2}\right]_0^b=\lim_{a\to-\infty}-\frac{1}{2}\left[1-e^{-a^2}\right]+\lim_{b\to\infty}-\frac{1}{2}\left[e^{-b^2}-1\right]=$$

$$=-\frac{1}{2}[1-0]+\left(-\frac{1}{2}\right)[0-1]=-\frac{1}{2}+\frac{1}{2}=0$$





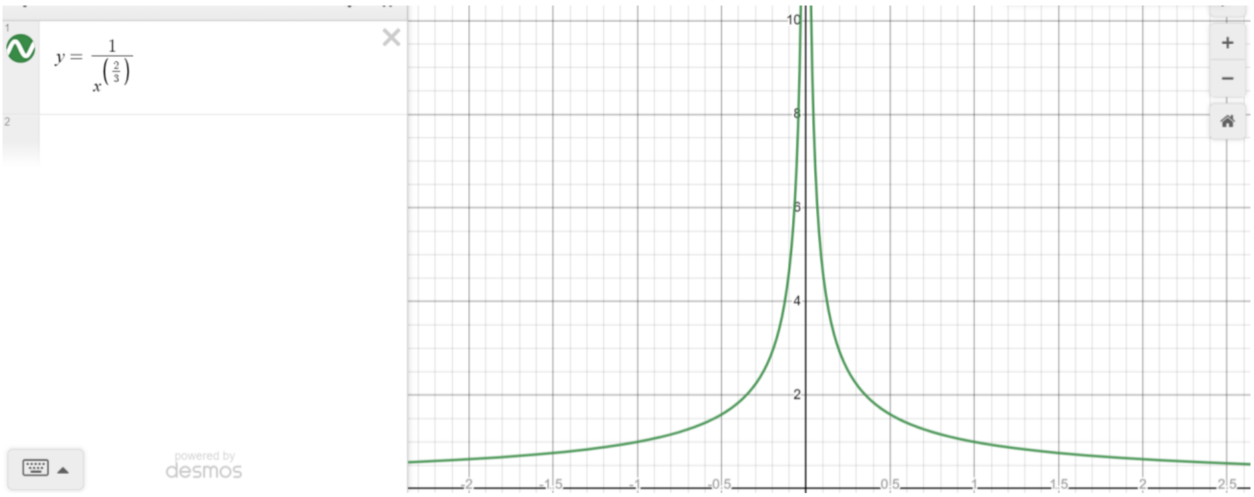
Example.

$$\int_{-1}^{1}\frac{1}{\sqrt[3]{x^2}}dx$$

Here, the endpoints are not the problem. The problem is inside the interval. Break the interval at the point of discontinuity (in this case 0) and evaluate each half.

$$\lim_{b \to 0} \int_{-1}^{b} x^{-\frac{2}{3}} dx + \lim_{a \to 0} \int_{a}^{1} x^{-\frac{2}{3}} dx = \lim_{b \to 0} \left[3x^{\frac{1}{3}} \right]_{-1}^{b} + \lim_{a \to 0} \left[3x^{\frac{1}{3}} \right]_{a}^{1} =$$

$$\lim_{b \to 0} 3b^{\frac{1}{3}} - 3(-1)^{\frac{1}{3}} + \lim_{a \to 0} 3(1)^{\frac{1}{3}} - 3a^{\frac{1}{3}} = -3(-1)^{\frac{1}{3}} + 3(1)^{\frac{1}{3}} = 3 + 3 = 6$$



In more complicated problems, you may need to use L'Hopital's rule to evaluate the limit.

For example:

$$\int_{0}^{\infty} xe^{-x} dx$$

End of Exam #1 material.

Review for the exam.