10/3/2022

Continue with Exact Equations
Numerical Methods: improvements on Euler's, Runge-Kutta
Review for the Exam

Continue with Exact Equations
Last time we talked about the procedure. Today we want to look at a test for exactness, and then we will look at integrating factors for making a problem into an exact equation.

Test for exactness.

$$M(x,y)dx + N(x,y)dy = 0$$

If the equation is exact then:

$$M_y = \frac{\partial M}{\partial y} = N_x = \frac{\partial N}{\partial x}$$

If this equation true, then the differential equation is exact.

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x} = f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$$

$$M = \frac{\partial f}{\partial x}, N = \frac{\partial f}{\partial y}$$

$$(2xy^2 + 4)dx + (2x^2y - 6)dy = 0$$

$$M(x,y) = 2xy^2 + 4, N(x,y) = 2x^2y - 6$$

$$M_y = 4xy, N_x = 4xy$$

This means the equation is exact and we can use the procedure we used last time to solve for the function.

Solve.

$$\int 2xy^2 + 4dx = x^2y^2 + 4x + g(y)$$

$$\int 2x^2y - 6dy = x^2y^2 - 6y + h(x)$$

$$f(x,y) = x^2y^2 + 4x - 6y + K$$

$$x^2y^2 + 4x - 6y + K = 0$$

Integrating factors for exact equations.
Sometimes a problem is not exact, but can be made exact by using an integrating factor (these problems usually result from variables canceling).

$$\mu(x) = e^{\int p(x)dx}$$

$$p(x) = \frac{(M_y - N_x)}{N}$$

If this function is a function of only x, then we can use this integrating factor.

$$\mu(y) = e^{\int q(y)dy}$$

$$q(y) = \frac{(N_x - M_y)}{M}$$

If this q function is only a function of y, then we can use this integrating factor.

Example.

$$(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)dx + (3x^2y^2 + 4y)dy = 0$$

$$M = 2xy^3 - 2x^3y^3 - 4xy^2 + 2x$$
$$M_y = 6xy^2 - 6x^3y^2 - 8xy$$

$$N = 3x^2y^2 + 4y$$
$$N_x = 6xy^2$$

$$M_y \neq N_x$$

This is not exact.

$$q(y) = \frac{(N_x - M_y)}{M} = \frac{6xy^2 - (6xy^2 - 6x^3y^2 - 8xy)}{2xy^3 - 2x^3y^3 - 4xy^2 + 2x} = \frac{6x^3y^2 + 8xy}{2xy^3 - 2x^3y^3 - 4xy^2 + 2x}$$
$$= \frac{2xy(3x^2y + 4)}{2x(y^3 - x^2y^3 - 2y^2 + 1)} = \frac{y(3x^2y + 4)}{(y^3 - x^2y^3 - 2y^2 + 1)}$$

This won't work.

$$p(x) = \frac{(M_y - N_x)}{N} = \frac{6xy^2 - 6x^3y^2 - 8xy - 6xy^2}{3x^2y^2 + 4y} = \frac{-6x^3y^2 - 8xy}{3x^2y^2 + 4y} = -\frac{2xy(3x^2y + 4)}{y(3x^2y + 4)} = -2x$$

$$\mu(x) = e^{\int p(x)dx}$$
$$\mu(x) = e^{\int -2x\,dx} = e^{-x^2}$$

Original:

$$(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)dx + (3x^2y^2 + 4y)dy = 0$$

Multiply by integrating factor

$$e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)dx + e^{-x^2}(3x^2y^2 + 4y)dy$$

$$M = e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)$$

$$M_y = e^{-x^2}(6xy^2 - 6x^3y^2 - 8xy)$$

$$N = e^{-x^2}(3x^2y^2 + 4y)$$
$$N_x = e^{-x^2}(-2x)(3x^2y^2 + 4y) + e^{-x^2}(6xy^2) = e^{-x^2}(-6x^3y^2 - 8xy + 6xy^2)$$

$$\int e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)dx$$

$$= \int (-2x)e^{-x^2}(-y^3 + x^2y^3 + 2y^2 - 1)dx =$$
$$u = (-y^3 + x^2y^3 + 2y^2 - 1), dv = \left(-2xe^{-x^2}\right)dx$$

$$du = 2xy^3dx, v = e^{-x^2}$$

$$e^{-x^2}(-y^3 + x^2y^3 + 2y^2 - 1) - \int 2xy^3\, e^{-x^2}dx = e^{-x^2}(-y^3 + x^2y^3 + 2y^2 - 1) + e^{-x^2} + g(y)$$

$$\int e^{-x^2}(3x^2y^2 + 4y)dy = e^{-x^2}(x^2y^3 + 2y^2) + h(x)$$

$$\int Mdx = -y^3e^{-x^2} + x^2y^3e^{-x^2} + 2y^2e^{-x^2} - e^{-x^2} + e^{-x^2} + g(y)$$

$$\int Ndy = x^2y^3e^{-x^2} + 2y^2e^{-x^2} + h(x)$$

$$f(x,y) = x^2y^3e^{-x^2} + 2y^2e^{-x^2} - e^{-x^2} + K$$

$$f_x = 2xy^3e^{-x^2} + x^2y^3(-2x)e^{-x^2} + 2y^2(-2x)e^{-x^2} - (-2x)e^{-x^2}$$

$$= e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x)$$

$$f_y = 3x^2y^2e^{-x^2} + 4ye^{-x^2} = e^{-x^2}(3x^2y^2 + 4y)$$

Generally only worry about these if the problem asks for it:
Determine if the equation is exact, and if it is not, find an appropriate integrating factor.

More Numerical approaches

Modified Euler': $y_{n+1} = y_n + hf\left[t_n + \frac{1}{2}h, y_n + \frac{1}{2}hf(t_n, y_n)\right]$
Takes the approximation used to estimate the slope at the midpoint of the interval.

$$x_0 = 1, y_0 = 2, \Delta x = 0.1$$
Use in Euler's method, estimate the slope at 1.05 and estimate the y-value at the midpoint, and then use that to estimate the slope over the whole interval. (as similar to the midpoint in Reimann sums)

Improved Euler's: (Trench, page 110)
Calculate the slope at both endpoints (1,2) and at (1.1, estimate using Euler's for y_{11}), and then average them. (more similar to the trapezoidal rule).

$$m_n = \frac{f\big(x_n, y_n(x_n)\big) + f\big(x_{n+1}, y(x_{n+1})\big)}{2}$$

$$y_{n+1} = y_n + m_n(\Delta x) = y_n + m_n h$$

Runge-Kutta has the idea to estimate at the initial endpoint, at the midpoint of the interval in two different ways, and then also at the other endpoint, and then produce a weighted average of all 4 values with the middle values being more highly weighted.

Runge-Kutta: $y_{n+1} = y_n + h\left(\frac{k_{n1}+2k_{n2}+2k_{n3}+k_{n4}}{6}\right)$,

$$k_{n1} = f(t_n, y_n), k_{n2} = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n1}\right),$$

$$k_{n3} = f\left(t_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n2}\right), k_{n4} = f(t_n + h, y_n + hk_{n3})$$

These are the end of Exam #1 material.

Review for the exam.
Solutions for first order differential equations: methods:

- Linear (integrating factor)
- Separable equations
- Exact
- Bernoulli – makes a non-linear into a linear
- Homogeneous – makes a non-separable, separable
- Exact integrating factor – makes a non-exact equation into an exact equation

Existence and uniqueness – (2.4?) – where is the differential equation defined, where does a solution exist, intervals of validity.

Euler's method
Runge-Kutta
Direction Fields
Autonomous Equations (only depend on the function variable $y' = y(y-1)$). They are always separable. They require partial fractions to solve.

Something super basic: test a solution in a differential equation to verify it is solution. Classifying differential equations by order, linearity, ordinary/partial.

Expect that at least one problem will have you solve for a constant.
In Bernoulli, Homogeneous and Exact Integrating Factor problems, I may only ask you to complete the problem part way: turn the Bernoulli into a linear (and then stop), homogeneous until it is separable, exact with integrating factor may have you stop when you can show that it is exact.

Application problem: probably 99% chance there will be a tank problem on the exam. Population exponential growth or decay, or Newton's Law of cooling, or similar.

Review the quizzes and written homeworks to understand my problem selection. The handouts also good.

Leave if you can't integrate as:

$$\int_{x_0}^{x} f(t)dt + C$$