10/31/2022

Reduction of Order
Review of Higher Order problems

Reduction of Order is a method to reduce the order of an ODE if you have already found one solution to the problem. Reduces a second order problem to a first order problem. Reduce a third order problem to a second order problem, etc. Especially helpful when the problem is not one of the standard type: not constant coefficient, and not Cauchy-Euler.

As long as you can find one initial solution to start (and we'll be given these solutions), then we can reduce the problem and find the remaining solution(s).

$$y'' + 4y' + 4y = 0$$

Reduction of order is typically applied to a homogeneous equation. Once both solutions are found, you can use variation of parameters (from the last lecture) to find the non-homogeneous solution.

From the characteristic equation: $k^2 + 4k + 4 = 0 \rightarrow (k+2)^2 = 0$, that the solution to this equation is when $k = -2$. We know one solution is $y = e^{-2t}$. We previously talked about how to obtain the second solution by multiplying by t to get $y = te^{-2t}$. We want to use reduction of order to show how we obtain this "trick".

The assumption reduction of order makes is that $y_2 = v(t)y_1(t)$
This trick is going to eliminate the part of the equation where the function is standing by itself, and therefore reducing the problem to only the higher-order terms.

In order to plug this assumption back into our equation, we need to take some derivatives.

$$y_2 = ve^{-2t}$$

$$y_2' = v'e^{-2t} - 2ve^{-2t}$$

$$y_2'' = v''e^{-2t} - 2v'e^{-2t} - 2v'e^{-2t} + 4ve^{-2t} = v''e^{-2t} - 4v'e^{-2t} + 4ve^{-2t}$$

Plug into original equation:

$$y'' + 4y' + 4y = 0$$

$$v''e^{-2t} - 4v'^{e^{-2t}} + 4ve^{-2t} + 4(v'e^{-2t} - 2ve^{-2t}) + 4ve^{-2t} = 0$$

$$v''e^{-2t} - 4v'^{e^{-2t}} + 4ve^{-2t} + 4v'e^{-2t} - 8ve^{-2t} + 4ve^{-2t} = 0$$

What must happen is that the v terms all cancel. (If they don't, there is an arithmetic mistake). Sometimes the v' terms will also cancel (but they don't have to).

If the v' terms don't all cancel, make a substitution to let $v' = u, and\ v'' = u'$ and then solve the resulting first order equation. (Typically, for textbook problems, this will be separation of variables or linear first order… but in the real world, may be other methods.)

In this case:

$$v''e^{-2t} = 0$$
$$v'' = 0$$

$$\int v'' = \int 0\, dt$$

$$v' = C$$

$$\int v' = \int C dt$$

$$v(t) = Ct + D$$

Typically, we will account for the constants in the general solution, so we don't need them here. We need the variable portion. $v(t) = t$

Our second solution is $y_2 = te^{-2t}$.

General solution: $y(t) = c_1 e^{-2t} + c_2 te^{-2t}$

If you use the constants in your $v(t)$? If $v(t) = c_1 t + c_2$

$$y(t) = (c_1 t + c_2)e^{-2t} = c_1 te^{-2t} + c_2 e^{-2t}$$

This the homogeneous solution, and if necessary we could then proceed to find the non-homogeneous solution if we needed one.

A more typical example of reduction of order is a problem that is not a constant coefficient nor a Cauchy-Euler problem.

Example.

$$(2x+1)y'' - 2y' - (2x+3)y = 0, y_1 = e^{-x}$$

Assume:

$$y_2 = ve^{-x}$$
$$y_2' = v'e^{-x} - ve^{-x}$$
$$y_2'' = v''e^{-x} - 2v'e^{-x} + ve^{-x}$$

$$(2x+1)(v''e^{-x} - 2v'e^{-x} + ve^{-x}) - 2(v'e^{-x} - ve^{-x}) - (2x+3)ve^{-x} = 0$$

$$2xv''e^{-x} - 4xv'e^{-x} + 2xve^{-x} + v''e^{-x} - 2v'e^{-x} + ve^{-x} - 2v'e^{-x} + 2ve^{-x} - 2xve^{-x} - 3ve^{-x} = 0$$

$$(e^{-x})(2xv'' - 4xv' + 2xv + v'' - 2v' + v - 2v' + 2v - 2xv - 3v) = 0$$

$$2xv'' - 4xv' + 2xv + v'' - 2v' + v - 2v' + 2v - 2xv - 3v = 0$$

$$v''(2x+1) + v'(-4x - 2 - 2) + v(2x + 1 + 2 - 2x - 3) = 0$$

$$(2x+1)v'' + (-4x-4)v' = 0$$

Make the substitution: $u = v', u' = v''$

$$u'(2x+1) + (-4x-4)u = 0$$

Separable.

$$u'(2x+1) = (4x+4)u$$

$$(2x+1)\frac{du}{dx} = (4x+4)\,u$$

$$\frac{du}{u} = \frac{4x+4}{2x+1}dx$$

$$\frac{du}{u} = \left(2 + \frac{2}{2x+1}\right)dx$$

$$\int \frac{1}{u}du = \int \left(2 + \frac{2}{2x+1}\right)dx$$

$$\ln(u) = 2x + \ln(2x+1) + C$$

$$\ln(u) = \ln(e^{2x}) + \ln(2x+1) + \ln(e^{C})$$

$$\ln(u) = \ln[e^{2x}(2x+1)A]$$

$$e^{C} = A$$
$$u = e^{2x}(2x+1)$$

$$u = v' = (2x+1)e^{2x}$$

Still need to find v. So we integrate again.

$$v = \int (2x+1)e^{2x}dx$$

$$u = 2x+1, dv = e^{2x}dx$$
$$du = 2du, v = \frac{1}{2}e^{2x}$$

$$v = (2x+1)\left(\frac{1}{2}e^{2x}\right) - 2\int \frac{1}{2}e^{2x}dx = \frac{1}{2}(2x+1)e^{2x} - \frac{1}{2}e^{2x}$$

What is y_2?

$$y_2 = \left(\frac{1}{2}(2x+1)e^{2x} - \frac{1}{2}e^{2x}\right)e^{-x} = \frac{1}{2}(2x+1)e^{x} - \frac{1}{2}e^{x} = xe^{x} + \frac{1}{2}e^{x} - \frac{1}{2}e^{x} = xe^{x}$$

$$y(x) = c_1 e^{-x} + c_2 x e^{x}$$

For a second order problem, you should have two unknown constants at the end for initial conditions.

A non-homogeneous example:

$$y'' - 2y' + y = 7x^{\frac{3}{2}}e^{x}, y_1 = e^{x}$$

$$y_2 = ve^{x}$$

$$y_2' = v'e^{x} + ve^{x}$$

$$y_2'' = v''e^{x} + 2v'e^{x} + ve^{x}$$

$$v''e^{x} + 2v'e^{x} + ve^{x} - 2(v'e^{x} + ve^{x}) + ve^{x} = 7x^{\frac{3}{2}}e^{x}$$

$$v''e^{x} + 2v'e^{x} + ve^{x} - 2v'e^{x} - 2ve^{x} + ve^{x} = 7x^{\frac{3}{2}}e^{x}$$

$$e^{x}(v'' + 2v' + v - 2v' - 2v + v) = 7x^{\frac{3}{2}}e^{x}$$
$$v'' + 2v' + v - 2v' - 2v + v = 7x^{\frac{3}{2}}$$

$$v'' = 7x^{\frac{3}{2}}$$

$$\int v'' = \int 7x^{\frac{3}{2}}dx$$

$$v' = 7\left(\frac{2}{5}\right)x^{\frac{5}{2}} + C$$

$$\int v' = \int \frac{14}{5}x^{\frac{5}{2}} + Cdx$$

$$v = \frac{14}{5}\left(\frac{2}{7}\right)x^{\frac{7}{2}} + Cx + D$$

$$v = \frac{4}{5}x^{\frac{7}{2}} + Cx$$

$$y_2 = \left(\frac{4}{5}\right)x^{\frac{7}{2}}e^{x} + Cxe^{x}$$

$$y(x) = \frac{4}{5}x^{\frac{7}{2}}e^{x} + c_1 x e^{x} + c_2 e^{x}$$

As a word of caution, if your first solution is a product already like $y_1 = e^x \sin(x)$, then your y_2 guess for the reduction of order becomes a triple product:

$$y_2 = v(x)e^x \sin(x)$$

$$(fgh)' = f'gh + fg'h + fgh'$$

$$(fg)'' = f''g + 2f'g' + fg''$$

Summary of Higher-Order ODEs

Higher Order problems have basically the same solution methods as second order problems.

Constant Coefficient Case:

$$ay''' + by'' + cy' + d = 0$$

Characteristic equation:

$$ak^3 + bk^2 + ck + d = 0$$

There is a formula for roots of cubic equations but it's nasty. That means that you will only be asked to solve problems that can be factored, or that one rational root (so that you can reduce the problem with long division—the quadratic factor that results can be done with the quadratic formula).

Cases: 3 real roots; 1 real root + 2 complex roots; repeated roots

Review factoring sum and difference of cubes, and factoring by grouping.
Graphing is the next best bet to look for rational roots you can divide out.

For fourth order, the formula from the coefficients is even nastier. Reduced to only being able to do problems we can factor or find rational roots graphically.

In general, sometimes you will see methods for finding complex roots of variables to a power. $k^5 = -1$. There are 5 solutions to this, and only one is real.

For fourth order cases, you can get repeated complex roots. The fix is the same. It's uncommon, but it is something to think about and look out for.

Similarly for Cauchy-Euler problems:

$$ax^3y''' + bx^2y'' + cxy' + dy = 0$$

$$y''' = (n)(n-1)(n-2)x^{n-3}$$

$$an(n-1)(n-2) + bn(n-1) + cn + d = 0$$

Is the auxiliary equation.

I will never go beyond the third order in Cauchy-Euler.

Non-homogenous cases:
Again, for the constant coefficient case, we have undetermined coefficients available. Our guesses (Ansatz) for $Y_p(x)$ basically work the same way. Mostly differences just come down to the time spent on algebra.

Variation of parameters: you need separate formula for each of the higher-order cases, and so (99% chance) I won't ask for using a variation of parameters on a third order or higher.

Abel's theorem: for finding the form of the Wronskian from the term of equation in standard form, the function multiplying the next lower derivative from the highest one. So if the nth derivative is the highest derivative, then the (n-1)st derivative, whatever function is multiplying that is p(x) for the formula.

$$y^{(n)} + p(x)y^{(n-1)} + q(x)y^{(n-2)} + \cdots + z(x)y = 0$$

If the term is missing, then p(x)=0.

$$y''' + 2xy'' + (4x-1)y' + xy = 0$$

$$p(x) = 2x$$

$$y''' + (2x-3)y' + x^2y = 0$$

$$p(x) = 0$$

Recall

$$W = c_0 e^{\int -p(x)dx}$$

When the ODE is in standard form.

Reminder:
If we are doing a third order problem, we need 3 solutions, and a 4-th order problem has 4 solutions, etc.

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = y_1 \begin{vmatrix} y_2' & y_3' \\ y_2'' & y_3'' \end{vmatrix} - y_2 \begin{vmatrix} y_1' & y_3' \\ y_1'' & y_3'' \end{vmatrix} + y_3 \begin{vmatrix} y_1' & y_2' \\ y_1'' & y_2'' \end{vmatrix}$$

Reduction of order can be applied if we can find one solution: but you still reduce to a 2nd order problem or a higher order problem with all the same restrictions. You may be able to apply reduction of order in stages.

Next time we'll talk about spring problems and charge problems (on circuit problems).
Then we'll review for the exam which 11/14.