12/5/2022

Systems of differential equations

Recall that

$$\frac{dy}{dt} = ky$$

$$\frac{dy}{y} = kdt$$

Integrate both sides to get the solution:

$$\ln y = kt + C$$
$$y = y_0 e^{kt}$$

Think of our parametric set of equations:

$$x(t), y(t)$$

$$\frac{dx}{dt} = ax$$
$$\frac{dy}{dt} = by$$

A little more complex:

$$\frac{dx}{dt} = ax + cy$$
$$\frac{dy}{dt} = by + dx$$

We turn our set of coefficients into a matrix, and variables into a vector.

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & c \\ d & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{dv}{dt} = Av$$

While we have to use matrix concepts to solve the equation, we end up with a solution that is of a similar form.

$$v(t) = v_0 e^{At}$$

We can represent this solution in terms of characteristics of the matrix. Characteristic properties of the matrix are eigenvalues and eigenvectors.

λ represents the eigenvalues (these are our powers of e)

The eigenvectors represent the relationship between x and y in our system.

The relationship systems of two differential equations and second order single variable problems:

Convert a second order problem into a system of equations?
The trick is to set $x' = y$ and then substitute.

$$x'' + 8x' + 12x = 0$$

If $x' = y, x'' = y'$

$$y' + 8y + 12x = 0$$
$$x' = y$$

$$x' = y$$
$$y' = -12x - 8y$$

$$\frac{dx}{dt} = 0x + 1y$$
$$\frac{dy}{dt} = -12x - 8y$$

We can turn this into a matrix, and so on.

$$A = \begin{bmatrix} 0 & 1 \\ -12 & -8 \end{bmatrix}$$

If we solve this in the second-order problem paradigm:

$$r^2 + 8r + 12 = 0$$
$$(r+6)(r+2) = 0$$
$$r = -6, -2$$
$$x(t) = c_1 e^{-6t} + c_2 e^{-2t}$$
$$y(t) = -6c_1 e^{-6t} - 2c_2 e^{-2t}$$

The idea is the characteristic value (the eigenvalues) are derived from the determinant of the matrix:

$$\begin{bmatrix} 0 - \lambda & 1 \\ -12 & -8 - \lambda \end{bmatrix}$$

Subtract the variable λ that we use for the eigenvalues off the diagonal, and then find the determinant.

$$(-\lambda)(-8 - \lambda) - (-12)(1) = 0$$

$$\lambda^2 + 8\lambda + 12 = 0$$

This is also called the characteristic equation.

$$\lambda = -6, -2$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} a \\ b \end{bmatrix} e^{-6t} + c_2 \begin{bmatrix} c \\ d \end{bmatrix} e^{-2t}$$

The eigenvectors are the relationships between the two solutions in the two equations.

For $\lambda = -6$
Plug into the matrix

$$\begin{bmatrix} 0-\lambda & 1 \\ -12 & -8-\lambda \end{bmatrix}$$

And solve the resulting system (it must be dependent)

$$\begin{bmatrix} 0-(-6) & 1 \\ -12 & -8-(-6) \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ -12 & -2 \end{bmatrix}$$

$$6a + b = 0$$
$$a = -\frac{1}{6}b$$
$$b = b$$

Let $b = -6$

$$a = 1$$
$$b = -6$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -6 \end{bmatrix} e^{-6t} + c_2 \begin{bmatrix} c \\ d \end{bmatrix} e^{-2t}$$

Do for the second eigenvalue.

For $\lambda = -2$
Plug into the matrix

$$\begin{bmatrix} 0-\lambda & 1 \\ -12 & -8-\lambda \end{bmatrix}$$

And solve the resulting system (it must be dependent)

$$\begin{bmatrix} 0-(-2) & 1 \\ -12 & -8-(-2) \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -12 & -6 \end{bmatrix}$$

$$2c + d = 0$$

$$c = -\frac{1}{2}d$$
$$d = d$$

Let $d = -2$

$$c = 1$$
$$d = -2$$

Put these values into my solution as the components of the second vector.

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -6 \end{bmatrix} e^{-6t} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-2t}$$

Any multiple of the eigenvectors will do. But it is standard practice to select values that are “small” whole numbers whenever possible.

We can say that linear second order problems can be converted to 2D systems that are also linear, as we just did, but we can also go backwards and convert most systems of 2D linear equations into a single second-order equation in one variable.

The solution methods just covered can be used without converting the system to a second-order problem.

Example.

$$\frac{dx}{dt} = 3x - 2y$$
$$\frac{dy}{dt} = 2x - 2y$$

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix}$$

Coefficient matrix is the key to finding the solutions.

1. Subtract λ off the diagonal
2. Find the determinant
3. Set equal to zero to find the characteristic equation.
4. Solve for the roots.
5. Find the eigenvectors for the solution by replacing λ in the matrix and solving the dependent system that results.
6. The general solution:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} a \\ b \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} c \\ d \end{bmatrix} e^{\lambda_2 t}$$

$$\begin{bmatrix} 3-\lambda & -2 \\ 2 & -2-\lambda \end{bmatrix}$$

$$(3-\lambda)(-2-\lambda) - (2)(-2) = 0$$

$$\lambda^2 - \lambda - 6 + 4 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0$$

$$\lambda = 2, -1$$

For $\lambda = 2$

$$\begin{bmatrix} 3-2 & -2 \\ 2 & -2-2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 2 & -4 \end{bmatrix}$$

$$a - 2b = 0$$
$$a = 2b$$

$$b = 1, a = 2 \rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

For $\lambda = -1$

$$\begin{bmatrix} 3-(-1) & -2 \\ 2 & -2-(-1) \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ 2 & -1 \end{bmatrix}$$

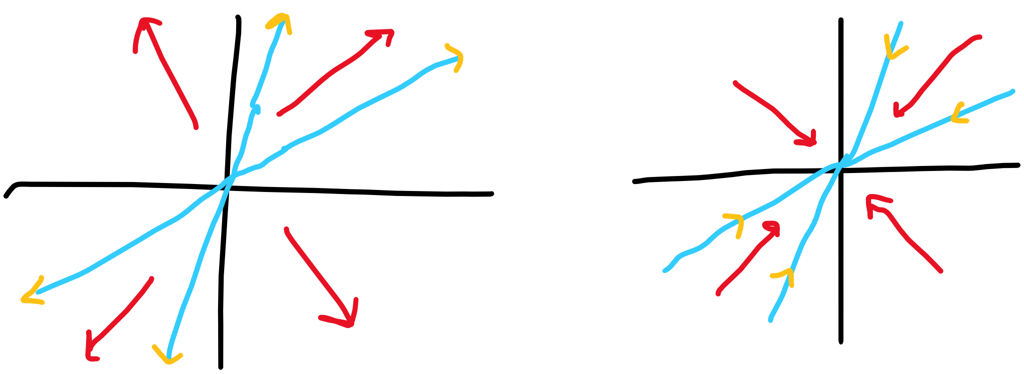
$$2c - d = 0$$
$$c = \frac{1}{2}d$$
$$d = d$$

$$d = 2, c = 1, \rightarrow \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

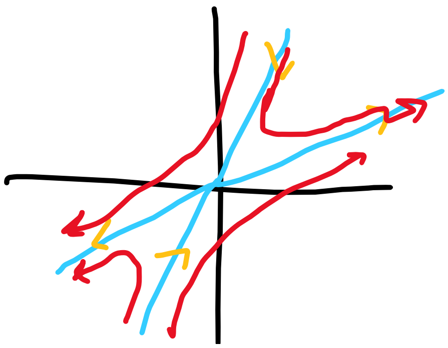
Solution:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{-t}$$

Suppose I have real roots to the characteristic equation (distinct). And there are two eigenvectors to go with each eigenvalue.



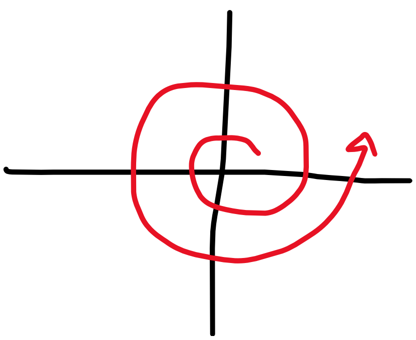
The origin repels the solutions when both eigenvalues are positive. (origin = repeller)
The origin attracts the solution when both eigenvalues are negative. (origin = attractor)

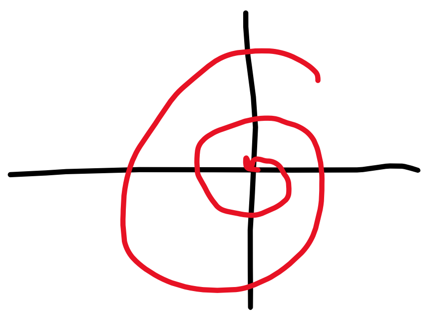


Here the positive eigenvalue grows, but the negative one shrinks… away to nothing. So even solutions are very similar to the pure negative solution, even with small coefficients, they will eventually get taken over by the larger eigenvalue and move away from the origin. The origin here is called a saddle point.

What if the solutions (the roots to the characteristic equation) are not real?

The oscillations we saw in the second-order problems translate to spirals in the system world. The real component determines whether the origin attracts (if it's negative) or if the origin repels (if it's positive). Because the roots are conjugates of each other, they have the same real part, and so there are no saddle points in the complex solution case.





Spiral out for positive real part of the root, and spiral in for negative real part of the root.
Can only be an attractor or a repeller.

Example with complex roots.

$$\vec{x}' = \begin{bmatrix} -7 & 10 \\ -4 & 5 \end{bmatrix} \vec{x}$$

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} -7 & 10 \\ -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A = \begin{bmatrix} -7 & 10 \\ -4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} -7-\lambda & 10 \\ -4 & 5-\lambda \end{bmatrix}$$

$$(-7-\lambda)(5-\lambda) - (-4)(10) = 0$$

$$\lambda^2 + 2\lambda - 35 + 40 = 0$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4(5)}}{2} = \frac{\left(-2 \pm \sqrt{-16}\right)}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$$\begin{bmatrix} -7 - (-1+2i) & 10 \\ -4 & 5 - (-1+2i) \end{bmatrix} = \begin{bmatrix} -6 - 2i & 10 \\ -4 & 6 - 2i \end{bmatrix}$$

$$-4a + (6 - 2i)b = 0$$
$$-4a = (-6 + 2i)b$$

$$a = \frac{-6 + 2i}{-4} b = \frac{3 - i}{2} b$$
$$b = b$$

Let $b = 2$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 - i \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} a \\ b \end{bmatrix} e^{\lambda_1 t} + c_2 \begin{bmatrix} c \\ d \end{bmatrix} e^{\lambda_2 t}$$

The problem we have here is that our eigenvalues are complex.

Recall:

$$e^{a+bi} = e^a\left(e^{bi}\right) = e^a(\cos b + i \sin b)$$

$$e^{(-1+2i)t} = e^{-t}(\cos 2t + i \sin 2t)$$

$$\begin{bmatrix} a \\ b \end{bmatrix} e^{\lambda_1 t} = \begin{bmatrix} 3 - i \\ 2 \end{bmatrix} e^{-t}(\cos 2t + i \sin 2t)$$

The algebra we need to work out for our two solutions is the other two pieces:

$$\begin{bmatrix} 3 - i \\ 2 \end{bmatrix} (\cos 2t + i \sin 2t)$$

FOIL to find real and imaginary components:

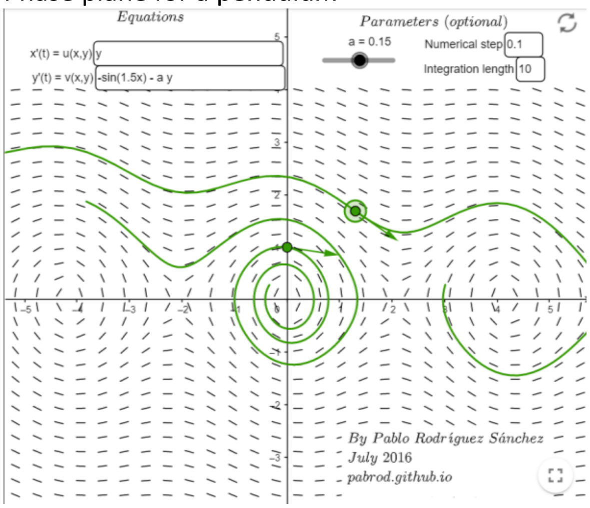
$$\begin{bmatrix} 3\cos 2t + 3i \sin 2t - i\cos 2t - i^2 \sin 2t \\ 2\cos 2t + 2i \sin 2t \end{bmatrix} = \begin{bmatrix} 3\cos 2t + \sin 2t \\ 2\cos 2t \end{bmatrix} + i \begin{bmatrix} 3\sin 2t - \cos 2t \\ 2\sin 2t \end{bmatrix}$$

Final general solution:

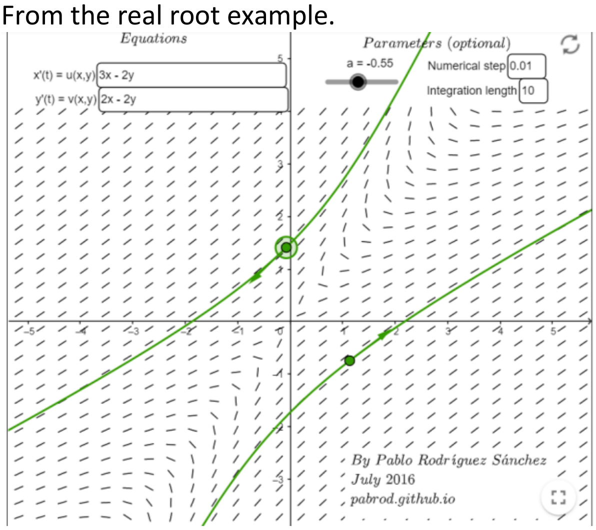
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 3\cos 2t + \sin 2t \\ 2\cos 2t \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 3\sin 2t - \cos 2t \\ 2\sin 2t \end{bmatrix} e^{-t}$$

Phase plane grapher: https://www.geogebra.org/m/utcMvuUy

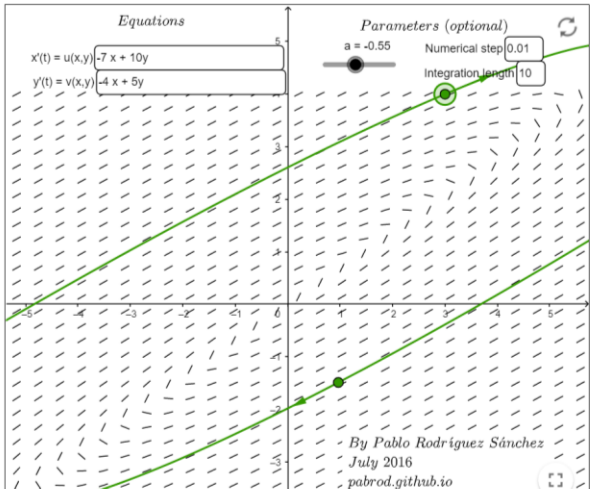
Phase plane for a pendulum



From the real root example.



Complex solution:



If you run into any repeated roots, you stop at that point.

Final Exam info:

Next Monday 12/12
Same format as previous exams. You'll have the whole day.

The only "new" material will come from Laplace transforms and what we talked about tonight. (the last 3 lectures).

For the new material, the best place to look for practice examples, is the quizzes and the handouts.

The comprehensive component will be similar to the last two exams, so those are the best place to look for review for those problems.