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Tank problems continued
Bernoulli equations
Homogeneous Equations

Concentration/Tank problems

$$\frac{dA}{dt} = Rate_{in} - Rate_{out}$$

Example.
Our tank originally contains 400 L of pure water. A mixture of 16g/L of sugar flows into the tank at a rate of 4 L/sec. The well-stirred mixture flows out at the same rate. Find an expression for the amount of sugar in the tank at any time t. And what amount of sugar is the equilibrium level?

$$\frac{dA}{dt} = Rate_{in} - Rate_{out}$$

$$Rate_{in} = \frac{16g}{L} \times \frac{4L}{s} = \frac{64g}{s}$$

$$Rate_{out} = \frac{A}{400L} \times \frac{4L}{s} = \frac{A}{100}\frac{g}{s}$$

$$\frac{dA}{dt} = 64 - \frac{A}{100}$$

$$\frac{dA}{dt} = -\frac{1}{100}(A - 6400)$$

$$\frac{dA}{A - 6400} = -\frac{1}{100}dt$$

$$\int \frac{dA}{A - 6400} = \int -\frac{1}{100}dt$$

$$\ln|A - 6400| = -\frac{1}{100}t + C$$

$$A - 6400 = e^{-\frac{1}{100}t+C} = A - 6400 = e^{-\frac{1}{100}t}e^{C} = A_0 e^{-\frac{1}{100}t}$$

$$A(t) = 6400 + A_0 e^{-\frac{1}{100}t}$$

For pure water, that means there is no sugar in the tank.

$$A(0) = 0$$

$$0 = 6400 + A_0 e^{-\frac{1}{100}(0)}$$
$$A_0 = -6400$$

$$A(t) = 6400 - 6400e^{-\frac{1}{100}t}$$

What is the equilibrium amount of sugar in the tank.

$$\lim_{t\to\infty} 6400 - 6400e^{-\frac{1}{100}t} = 6400$$

6400 g of sugar.

This is an example of same rate in/same rate out.

Example.
Our tank is 1000 L, and originally contains 400 L of pure water. A mixture of 16g/L of sugar flows into the tank at a rate of 4 L/sec. The well-stirred mixture flows out at the rate of 2 L/sec. Find an expression for the amount of sugar in the tank at any time t. How long until the tank overflows? What is the amount of sugar in the tank at that time?

$$\frac{dA}{dt} = Rate_{in} - Rate_{out}$$

$$Rate_{in} = \frac{16g}{L} \times 4\frac{L}{s} = 64\frac{g}{s}$$

$$Rate_{out} = \frac{A}{400+2t} \times \frac{2L}{s} = \frac{2A}{400+2t}\frac{g}{s} = \frac{A}{200+t}$$

$$\frac{dA}{dt} = 64 - \frac{A}{200+t}$$

$$A' + \left(\frac{1}{200+t}\right)A = 64$$

$$\mu = e^{\int \frac{1}{200+t}dt} = e^{\ln|200+t|} = 200+t$$

$$(200+t)A' + A = 64(200+t)$$

$$[(200+t)A]' = 64(200+t)$$

$$(200+t)A = 64\int 200 + tdt = 64\left[200t + \frac{1}{2}t^2\right] = 12800t + 32t^2 + C$$

$$(200+t)A = 12800t + 32t^2 + C$$

$$A = \frac{12800t + 32t^2 + C}{200+t}$$

$$(200+t)A = 64\int 200+t dt$$

$$u = 200+t$$
$$du = dt$$
$$64\int u du = 64\left(\frac{1}{2}u^2\right)+C = \frac{64}{2}(200+t)^2+C$$

$$(200+t)A = 32(200+t)^2+C$$

$$A = \frac{32(200+t)^2}{200+t}+\frac{C}{200+t}$$

$$A(t) = 32(200+t)+\frac{C}{200+t}$$

$$A(0) = 0$$
$$0 = 32(200+0)+\frac{C}{200+0}$$
$$-6400 = \frac{C}{200}$$

$$C = -1280000 = -1.28\times 10^6$$

$$A(t) = 32(200+t)-1.28\times\frac{10^6}{200+t}$$

Amount of time until the tank overflows $\frac{1000-400}{2} = 300\ s$

$$A(300) = 32(200+300)-1.28\times\frac{10^6}{200+300} = 13{,}440\ g$$

Bernoulli equations.
These are non-linear equations but they look somewhat similar to linear equations. And they can be transformed into linear equations with an appropriate substitution/integrating factor.

$$y' + p(t)y = f(t)y^n$$

The basic process involves multiplying the whole equation by something (integrating factor), and then make a substitution, which will make the equation a linear equation.

1. $(1-n)y^{-n}$ multiply whole equation by this expression
   n is a constant from the right side of the equation, the power of y
2. Make a substitution $z = y^{1-n}, \frac{dz}{dt} = (1-n)y^{-n}$

At that point the equation will be linear.

Example.

$$\frac{dy}{dt} + 2ty = ty^2$$

$$y' + 2ty = ty^2$$

1. $n = 2, (1-2)y^{-2} = -1y^{-2} = -y^{-2}$
   Multiply the whole equation by this expression.

$$-y^{-2}y' - 2ty(y^{-2}) = -t$$

$$-y^{-2}y' - 2t(y^{-1}) = -t$$

2. The substitute for the y variable where the linear term used to be.

$$z = y^{-1}$$

$$z' = \frac{dz}{dt} = -1(y^{-2})y' = -y^{-2}y'$$

$$z' - 2tz = -t$$

This equation is now linear.

$$\mu = e^{\int -2t dt} = e^{(-t^2)}$$

$$e^{-t^2}z' - 2te^{-t^2}z = -te^{-t^2}$$

$$\left(e^{-t^2}z\right)' = -te^{-t^2}$$

$$e^{-t^2}z = \int -te^{-t^2}dt$$

$$u = -t^2, du = -2tdt, \frac{1}{2}du = -tdt$$

$$e^{-t^2}z = \frac{1}{2}e^{-t^2} + C$$

$$e^{t^2}e^{-t^2}z = e^{t^2}\left(\frac{1}{2}e^{-t^2} + C\right)$$

$$z = \frac{1}{2} + Ce^{t^2}$$

$$\frac{1}{y} = \frac{1}{2} + Ce^{t^2}$$

You can solve this for y if you want, but that isn't necessary (some problems will be more difficult).

Homogeneous equations
Converts into separable equations.

One version is solved for $\frac{dy}{dx} = f(x,y) = -\frac{M(x,y)}{N(x,y)}$
One version is in differential form $M(x,y)dx + N(x,y)dy = 0$

If problem is homogeneous: If we replace $y \to ty, and\ x \to tx$, then the t's will cancel out (in the first form), or completely factor out (in the differential form). The degree of the homogeneous equation is described in terms of the power of t that comes out.

This tends to mean that both M and N are polynomials of the same degree.

Example.

$$(x-y)dx + xdy = 0$$

$$xdy = (y-x)dx$$
$$\frac{dy}{dx} = \frac{y-x}{x}$$

Test to see if it's homogeneous.

$$\frac{ty-tx}{tx} = \frac{t(y-x)}{(t)(x)} = \frac{y-x}{x}$$

Since I cancelled one power of t, this is homogeneous, degree 1.

$$(tx-ty)dx + txdy = 0$$

$$t(x-y)dx + t(x)dy = 0$$
$$t[(x-y)dx + xdy] = 0$$

This is homogeneous degree 1 because I factored out one power of t.

Trick here is to substitute $y = vx$
v is a function of x, and the extra x will cancel, and leave me with an equation in v which is separable.

$$y' = v'x + v$$

$$\frac{dy}{dx} = \frac{y-x}{x}$$

$$v'x + v = \frac{vx-x}{x} = \frac{x(v-1)}{x} = v-1$$

$$v'x + v = v - 1$$

$$v'x = -1$$

$$\frac{dv}{dx}x = -1$$

$$dv = -\frac{1}{x}dx$$

$$\int dv = \int -\frac{1}{x}dx$$

$$v = -\ln|x| + C$$

$$y = vx \rightarrow v = \frac{y}{x}$$

$$\frac{y}{x} = -\ln|x| + C$$

$$y = -x\ln|x| + Cx$$

More examples.

Bernoulli example.

$$y' + \frac{y}{x} = x\sqrt{y}$$

1. $(1-n)y^{-n}$

$$n = \frac{1}{2}$$

$$y' + \frac{y}{x} = xy^{\frac{1}{2}}$$

$$\left(1-\frac{1}{2}\right)y^{-\left(\frac{1}{2}\right)} = \frac{1}{2}y^{-\left(\frac{1}{2}\right)}$$

$$\frac{1}{2}y^{-\left(\frac{1}{2}\right)}y' + \left(\frac{1}{x}\right)y\frac{1}{2}y^{-\left(\frac{1}{2}\right)} = xy^{\frac{1}{2}}\frac{1}{2}y^{-\left(\frac{1}{2}\right)}$$

$$\frac{1}{2}y^{-\left(\frac{1}{2}\right)}y' + \left(\frac{1}{x}\right)\frac{1}{2}y^{1/2} = \frac{1}{2}x$$

2. $z = y^{\frac{1}{2}}, z' = \frac{1}{2}y^{-\frac{1}{2}}y'$

$$z' + \frac{1}{2x}z = \frac{1}{2}x$$

$$\mu = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\int \frac{1}{x}dx} = e^{\frac{1}{2}\ln x} = e^{\ln\sqrt{x}} = \sqrt{x} = x^{\frac{1}{2}}$$

$$x^{\frac{1}{2}}z' + \frac{1}{2x}x^{\frac{1}{2}}z = \frac{1}{2}x\left(x^{\frac{1}{2}}\right)$$

$$x^{\frac{1}{2}}z' + \frac{1}{2}x^{-\frac{1}{2}}z = \frac{1}{2}x^{\frac{3}{2}}$$

$$\left(x^{\frac{1}{2}}z\right)' = \frac{1}{2}x^{\frac{3}{2}}$$

$$x^{\frac{1}{2}}z = \int \frac{1}{2}x^{\frac{3}{2}}dx = \frac{1}{2}\left(\frac{2}{5}\right)x^{\frac{5}{2}} + C$$

$$x^{\frac{1}{2}}z = \frac{1}{5}x^{\frac{5}{2}} + C$$

$$z = \frac{1}{5}x^2 + Cx^{-\frac{1}{2}}$$

$$y^{\frac{1}{2}} = \frac{1}{5}x^2 + Cx^{-\frac{1}{2}}$$

Homogeneous example.

$$y' = \frac{xy}{x^2 - y^2}$$

$$y = vx, y' = v'x + v$$

$$v'x + v = \frac{x(vx)}{x^2 - (vx)^2} = \frac{x^2(v)}{x^2(1 - v^2)} = \frac{v}{1 - v^2}$$

$$v'x + v = \frac{v}{1 - v^2}$$

$$v'x = \frac{v}{(1 - v^2)} - v$$

$$v'x = \frac{v}{(1 - v^2)} - \frac{v(1 - v^2)}{1 - v^2} = \frac{v - v(1 - v^2)}{1 - v^2} = \frac{(v - v + v^3)}{1 - v^2} = \frac{v^3}{1 - v^2}$$

$$\frac{dv}{dx}(x) = \frac{v^3}{1 - v^2}$$

$$\frac{(1 - v^2)dv}{v^3} = \frac{1}{x}dx$$

$$\left(\frac{1}{v^3} - \frac{v^2}{v^3}\right) dv = \frac{1}{x} dx$$

$$\left(v^{-3} - \frac{1}{v}\right) dv = \frac{1}{x} dx$$

$$\int \left(v^{-3} - \frac{1}{v}\right) dv = \int \frac{1}{x} dx$$

$$= -\frac{1}{2} v^{-2} - \ln|v| = \ln|x| + C$$

$$y = vx \rightarrow v = \frac{y}{x}$$

$$-\frac{1}{2}\left(\frac{y}{x}\right)^{-2} + \ln\left|\frac{y}{x}\right| = \ln|x| + C$$

$$-\frac{1}{2}\left(\frac{x}{y}\right)^{2} + \ln\left|\frac{y}{x}\right| = \ln|x| + C$$

$$-\frac{x^2}{2y^2} + \ln|y| - \ln|x| = \ln|x| + C$$

$$-\frac{x^2}{2y^2} + \ln|y| = 2\ln|x| + C$$

Exact Equations.

$$(2xy - 9x^2)dx + (2y + x^2 + 1)dy = 0$$

$$M(x, y) = 2xy - 9x^2$$

$$N(x, y) = 2y + x^2 + 1$$

There is a function ϕ (or ψ) of both x and y that if you take the partial derivative of the function with respect to x, you get M(x,y), and if you take the partial derivative of the function with respect to y, you get N(x,y).

A partial derivative is a derivative of a multivariable function where you treat all variables except the one you are interested in as constants. So if you have a function $f(x, y)$, and you take the derivative with respect to x, $\frac{\partial f}{\partial x}$, then the y-variable is treated as a constant, and you take the derivative of x "normally".

$$f(x, y) = x^2 + 2xy - y^3 + ye^{3x}$$

$$\frac{\partial f}{\partial x} = 2x + 2y(1) - 0 + y(3e^{3x}) = 2x + 2y + 3ye^{3x}$$

$$\frac{\partial f}{\partial y} = 0 + 2x(1) - 3y^2 + e^{3x}(1) = 2x - 3y^2 + e^{3x}$$

The same principles apply in the anti-derivative. We treat all other variables as constant, and only work on one at a time.

$$\int 2x + 2y + 3ye^{3x}dx = \int 2xdx + \int 2ydx + \int 3ye^{3x}dx = \int 2xdx + 2y\int dx + y\int 3e^{3x}dx$$

$$= x^2 + 2y(x) + y(e^{3x}) + g(y)$$

I can't recover any constants, but also I can't recover any terms that contained only y (since I was integrating with x).

$$\int 2x - 3y^2 + e^{3x}dy = \int 2xdy - \int 3y^2dy + \int e^{3x}dy = 2x\int dy - \int 3y^2dy + e^{3x}\int dy =$$

$$2x(y) - y^3 + e^{3x}(y) + h(x)$$

What is f(x,y) from this process?

Any terms that contain both x and y are in f(x,y), and any terms with one variable with only appear in one of the antiderivatives.

	$f(x,y) =$
In both antiderivatives	$2xy + ye^{3x}$
In the x-only antiderivative	$+x^2$
In the y-only antiderivative	$-y^3$

$$f(x,y) = 2xy + ye^{3x} + x^2 - y^3 + K$$

I can't recover the constant unless I have an initial condition.

$$2xy + ye^{3x} + x^2 - y^3 + K = 0$$

Back to our problem:

$$(2xy - 9x^2)dx + (2y + x^2 + 1)dy = 0$$

Treat $M(x,y) = \frac{\partial f}{dx}$ and $N(x,y) = \frac{\partial f}{\partial y}$.

$$\int M(x,y)dx = \int 2xy - 9x^2dx = y\int 2xdx - 9\int x^2dx = x^2y - \frac{9}{3}x^3 + g(y)$$

$$\int N(x,y)dy = \int 2y + x^2 + 1dy = \int 2y\,dy + x^2 \int dy + \int 1dy = y^2 + x^2y + y + h(x)$$

$$f(x,y) = x^2y - 3x^3 + y^2 + y + K$$

$$x^2y - 3x^3 + y^2 + y + K = 0$$

So this implicit function satisfies our differential equation.

Next time, we'll do another example. And we'll look at integrating factors that can make some equations into exact equations even if they aren't to start with.

Return to numerical methods (Runge-Kutta).