MTH 267, Quiz #11, Fall 2022 Name KEY

Instructions: Show all work. Answers without work required to obtain the solution will not receive full credit. Some questions may contain multiple parts: be sure to answer all of them. Give exact answers unless specifically asked to estimate.

1. Rewrite $u'' + \frac{1}{2}u' + 2u = 0$ as a system of first order equations.

$u = x$
$y = u' = x'$
$y' = u''$

$x' = y$
$y' = -2x - \frac{1}{2}y$

$y' + \frac{1}{2}y + 2x = 0 \rightarrow y' = -2x - \frac{1}{2}y$

$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -2 & -1/2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

2. Rewrite the system $\begin{cases} x_1' = 3x_1 - 2x_2 \\ x_2' = 2x_1 - 2x_2 \end{cases}$ as a single second order equation.

$2x_1 = x_2' - 2x_2$

$x_1 = x$
$x_2 =$

$x_1' = 3x_1 - 2x_2$
$x_1' - 3x_1 = -2x_2$
$-\frac{1}{2}x_1' + \frac{3}{2}x_1 = x_2$
$x_2' = -\frac{1}{2}x_1'' + \frac{3}{2}x_1'$

$-\frac{1}{2}x_1'' + \frac{3}{2}x_1' = 2x_1 - 2(-\frac{1}{2}x_1' + \frac{3}{2}x_1)$
$-\frac{1}{2}x_1'' + \frac{3}{2}x_1' = 2x_1 + 2x_1' - 3x_1$
$-\frac{1}{2}x_1'' + \frac{3}{2}x_1' = -x_1 + 2x_1'$
$-\frac{1}{2}x_1'' - \frac{1}{2}x_1' + x_1 = 0 \rightarrow (-2) = x_1'' + x_1' - 2x_1 = 0 \rightarrow \boxed{x'' + x' - 2x = 0}$

3. Find the eigenvalues and eigenvectors of

a. $A = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}$

$(5-\lambda)(1-\lambda) + 3 = 0$
$\lambda^2 - 6\lambda + 5 + 3 = 0$
$\lambda^2 - 6\lambda + 8 = 0$
$(\lambda-4)(\lambda-2) = 0$
$\lambda = 4, 2$

$\lambda = 4$
$\begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$
$x_1 - x_2 = 0$
$x_1 = x_2$
$x_2 = x_2$
$\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 2$
$\begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}$
$3x_1 - x_2 = 0$
$x_1 = 1/3 x_2$
$x_2 = x_2$
$\vec{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

b. $B = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix}$

$(-2-\lambda)(2-\lambda) - 1 = 0$
$\lambda^2 + 2\lambda - 2\lambda - 4 - 1 = 0$
$\lambda^2 - 5 = 0$
$\lambda = \pm\sqrt{5}$

$\lambda = \sqrt{5}$
$\begin{pmatrix} -2-\sqrt{5} & 1 \\ 1 & 2-\sqrt{5} \end{pmatrix}$
$x_1 + (2-\sqrt{5})x_2 = 0$
$x_1 = (\sqrt{5}-2)x_2$
$x_2 = x_2$

$\vec{v}_1 = \begin{pmatrix} -2+\sqrt{5} \\ 1 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} -2-\sqrt{5} \\ 1 \end{pmatrix}$

4. Solve $\vec{x}' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix}\vec{x}, \vec{x}(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ [Hint: see #3a]

$\vec{x} = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{2t}$

$c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}(1) + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}(1) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\cancel{c_1} + c_2 = 2$
$-\cancel{c_1} \mp 3c_2 = +1$

$-2c_2 = 3$
$c_2 = -3/2$
$c_1 - 3/2 = 2$
$c_1 = 7/2$

$\vec{x} = 7/2\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{4t} + c_2\begin{pmatrix} 1 \\ 3 \end{pmatrix}e^{2t}$

5. Draw a direction field for the system $\vec{x}' = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} \vec{x}$ and plot several trajectories of the system.

$$(3-\lambda)(-2-\lambda)+4=0$$
$$\lambda^2-\lambda-6+4=0$$
$$\lambda^2-\lambda-2=0$$
$$(\lambda-2)(\lambda+1)=0$$

$\lambda=2,-1$

$\lambda=2$ $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$

$x_1-2x_2=0$

$x_1=2x_2$

$x_2=x_2$

$\hat{v}_1=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\lambda=-1$

$\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$

$2x_1-x_2=0$

$x_1=\frac{1}{2}x_2$

$x_2=x_2$

$\vec{v}_2=\begin{pmatrix} 1 \\ 2 \end{pmatrix}$