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MTH 264, Exam #2, Fall 2023 Name KEY

Instructions: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error of any kind.

1. For each of the series below, determine whether the series converges or diverges (in #6, you'll be asked to prove your conclusion, so it may help to do those problems first/together). (6 points each)
   a. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$ alternating series test (See #6)
   Converges
   b. $\sum_{n=0}^{\infty} \frac{1}{n!}$ Converges
   c. $\sum_{n=0}^{\infty} \frac{2^n+1}{5^n+1}$ Converges
   d. $\sum_{n=2}^{\infty} \frac{\ln n}{n^4}$ Converges
   e. $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$ Converges
   f. $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2+1}$ Converges
   g. $\sum_{n=0}^{\infty} e^{-n}$ Converges

h. $\sum_{n=1}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)$ converges

i. $\sum_{n=0}^{\infty}\frac{3^n}{n!}$ Converge

j. $\sum_{n=1}^{\infty}\left(\frac{4n}{5n+3}\right)^n$ Converges

2. Find N such that $R_N \leq 10^{-5}$, for the convergent series. (10 points)

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}$$

$\frac{1}{(n+1)^3} < 10^{-5} \rightarrow (n+1)^3 > 10^{+5}$

$n+1 > 46.4 \Rightarrow 47$

$n = 46$

3. For the sequence below. i) Determine if the sequence is monotonic (or is monotonic after some finite value of n). You may determine this graphically or by calculating derivatives. ii) Determine the bounds (above and below of the sequence). iii) Can you apply the bounded & monotonic theorem for convergence to this sequence? iv) Does this sequence converge by another theorem? If so, which one? v) If the sequence converges, what does it converge to? (20 points)

$$a_n = \left(-\frac{2}{3}\right)^n$$

i) the sequence is alternating, so not monotonic.

ii) bounded by 1 above and -1 below for $n \geq 0$

iii) it is not monotonic, but if we consider even and odd terms separately then we could apply the bounded & monotonic theorem since all odd terms are negative and increasing, and even ones all positive and decreasing.

iv) geometric sequence w/ $r < 1$

v) terms in sequence go to 0

4. Use a power series to approximate the integral $\int_0^1 \frac{e^{-x}}{x}dx$. Use 6 terms, given that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Round your answer to 4 decimal places. (10 points)

$e^{-x} = \sum_{n=0}^{\infty} \frac{x^n(-1)^n}{n!}$ $\quad \frac{e^{-x}}{x} = \sum \frac{x^{n-1}(-1)^n}{n!}$

$\int_0^1 \sum_{n=0}^{\infty} \frac{(-1)^n x^{n-1}}{n!}dx = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n\ \ )!n}\Big|_0^1 = \sum_{n=0}^{\infty} \frac{(-1)^n 1^n}{n! \cdot n}$

$\approx \frac{(-1)^0(1)}{(1)(0)} + \frac{(-1)^1(1)}{(1)(1)} + \frac{(-1)^2(1)}{(2)(2)} + \frac{(-1)^3(1)}{(6)(3)} + \frac{(-1)^4(1)}{(24)(4)} + \frac{(-1)^5(1)}{(120)(5)}$ does not converge (at 0)

undefined

5. What is the maximum error R_n for the Taylor polynomial $-\ln(1-x) \approx x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4}$ on the interval $\left[0, \frac{1}{2}\right]$. (9 points)

$f' = \frac{1}{1-x} = (1-x)^{-1}$

$f'' = (1-x)^{-2}$

$f''' = 2(1-x)^{-3}$

$f^{IV} = 6(1-x)^{-4}$

$f^{V} = 24(1-x)^{-5} = \frac{24}{(1-x)^5}$

$\frac{24}{(1-0)^5} = 24$

$\frac{24}{(1-1/2)^5} = \frac{24}{(1/2)^5} =$

$24(32) = 768$ max

$R_4 : \frac{768}{5!} = \frac{768}{120} \approx 6.4$

Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

6. For each of the series below (same as in #1), state the name of the test used to determine convergence. Show the work here to support your conclusion above. (8 points each)

a. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\ln(n+1)}$ $\quad \lim_{n\to\infty} \left| \frac{1}{\ln(n+1)} \right| = 0$ converges by alternating series test (conditional)

b. $\sum_{n=0}^{\infty} \frac{1}{n!}$ $\quad \lim_{n\to\infty} \frac{(n!)}{(n+1)!} = \frac{1}{n+1} = 0 < 1$ converges by ratio test

c. $\sum_{n=0}^{\infty} \frac{2^n+1}{5^n+1}$ limit comparison w/ $\frac{2^n}{5^n} = \left(\frac{2}{5}\right)^n \rightarrow$ converges by geometric series test

$\lim_{n\to\infty} \frac{2^n+1}{5^n+1} \cdot \frac{5^n}{2^n} = \quad = 1$ converges

d. $\sum_{n=2}^{\infty} \frac{\ln n}{n^4}$ direct comparison w/ $\frac{n}{n^4} = \frac{1}{n^3}$ since $\frac{\ln n}{n^4} \leq \frac{n}{n^4}$

and $\frac{1}{n^3}$ converges by p series

$p > 1$

e. $\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$ converges by p-series test $p > 1$

f. $\sum_{n=1}^{\infty} \frac{\arctan n}{n^2+1}$ $\int_1^{\infty} \frac{\arctan x}{x^2+1}\,dx = \frac{1}{2}(\arctan x)^2 \Big|_1^{\infty} = \frac{1}{2}\left[\left(\frac{\pi}{2}\right)^2 - \left(\frac{\pi}{4}\right)^2\right]$

finite

Converges by integral test

g. $\sum_{n=0}^{\infty} e^{-n}$ $(e^{-1}) < 1$ converges by geometric series test

h. $\sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$ $\lim_{n\to\infty} \frac{1}{n+1} = 0$ converges by telescoping series test

i. $\sum_{n=0}^{\infty} \frac{3^n}{n!}$ converges by ratio test $\lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \frac{3}{n+1} = 0 < 1$

j. $\sum_{n=1}^{\infty} \left(\frac{4n}{5n+3}\right)^n$ converges by the root test $\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{4n}{5n+3}\right)^n\right|} = \lim_{n\to\infty} \frac{4n}{5n+3} = \frac{4}{5} < 1$

7. For the sequence $1, \frac{4}{3}, \frac{3}{2}, \frac{8}{5}, \ldots$, find a formula for the nth term of the sequence (starting at n=0). [Hint: try rewriting the first and third terms.] (10 points)

$\frac{2}{2}, \frac{4}{3}, \frac{6}{4}, \frac{8}{5} \cdots$ $\frac{2(1)}{0+2}, \frac{2(2)}{1+2}, \frac{2(3)}{2+2}, \frac{2(4)}{3+2} \cdots$

$\frac{2(n+1)}{n+2}$ $\frac{2(n+1)}{n+2} = a_n$

8. Find the interval of convergence of the power series. (10 points each)

$$\sum_{n=0}^{\infty} \frac{n!\,x^n}{(2n)!}$$

$$\lim_{n\to\infty} \frac{(n+1)!\,x^{n+1}}{(2n+2)!} \cdot \frac{(2n)!}{n!\,x^n} = \lim_{n\to\infty} \frac{(n+1)\,x}{(2n+1)(2n+2)} = \lim_{n\to\infty} \frac{x\,\cancel{(n+1)}}{(2n+1)2\cancel{(n+1)}} = \lim_{n\to\infty} \frac{x}{2(2n+1)} = 0$$

Converges on $(-\infty, \infty)$

9. Find the Taylor Polynomial for the function at the indicated value of c. Use the tables provided. (15 points)

$$f(x) = \ln(x), n = 5, c = 2$$

n	n!	$f^{(n)}(x)$	$f^{(n)}(c)$	$(x-c)^n$	$\frac{f^{(n)}(c)}{n!}(x-c)^n$
0	1	$\ln x$	$\ln 2$	1	$\ln 2$
1	1	$1/x$	$\frac{1}{2}$	$x-2$	$\frac{1}{2}(x-2)$
2	2	$-1/x^2$	$\frac{-1}{4}$	$(x-2)^2$	$-\frac{1}{4}\cdot\frac{1}{2}(x-2)^2$
3	6	$\frac{2}{x^3}$	$\frac{2}{8}=\frac{1}{4}$	$(x-2)^3$	$\frac{1}{4}\cdot\frac{1}{6}(x-2)^3$
4	24	$\frac{-6}{x^4}$	$\frac{-6}{16}=\frac{-3}{8}$	$(x-2)^4$	$-\frac{3}{8}\cdot\frac{1}{24}(x-2)^4$
5	120	$\frac{24}{x^5}$	$\frac{24}{32}=\frac{3}{4}$	$(x-2)^5$	$\frac{3}{4}\cdot\frac{1}{120}(x-2)^5$
6	720	$\frac{-120}{x^6}$	$\frac{-120}{64}=\frac{-15}{8}$	$(x-2)^6$	

$P_n(x) = \ln 2 + \frac{1}{2}(x-2) - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \frac{1}{64}(x-2)^4 + \frac{1}{160}(x-2)^5$

10. Find the power series for the functions below. Write your answers with the sum starting at n=0. (12 points each)

a. $f(x) = \arctan x$

$f' = \frac{1}{1+x^2}$ $\quad r=(-x^2)\quad a=1$

$\sum_{n=0}^{\infty} (-1)^n x^{2n} \rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)}$

b. $g(x) = \frac{x^2}{(1-4x)^3}$

$a(1-r)^{-1} = \sum ar^n$

$a(1-r)^{-2} = \sum_{n=1}^{\infty} anr^{n-1} =$

$2a(1-r)^{-3} = \sum_{n=2}^{\infty} an(n-1)r^{n-2} \rightarrow \sum_{n=0}^{\infty} a(n+2)(n+1)r^n = \frac{2a}{(1-r)^3}$

$r = +4x \quad a = \frac{1}{2}x^2$

$\sum_{n=0}^{\infty} (\frac{1}{2}x^2)(n+2)(n+1)(4x)^n =$

$\sum_{n=0}^{\infty} (\frac{1}{2})\underset{2^{2n}}{4^n}(n+2)(n+1)x^2x^n =$

$\sum_{n=0}^{\infty} 2^{2n-1}(n+2)(n+1)x^{n+2}$