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MTH 264, Final Exam, Fall 2023 Name KEY

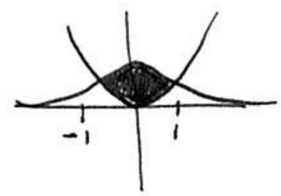
Instructions: Work the problems below as directed. Show all work. Clearly mark your final answers. Use exact values unless the problem specifically directs you to round. Simplify as much as possible. Partial credit is possible, but solutions without work will not receive full credit.

Part 1: These questions you will submit answers to in Canvas. Show all work and submit the work with Part 2 of the exam. But you must submit the answers in Canvas to receive credit. Each question/answer will be listed separately. The Canvas question will refer to the number/part to indicate where you should submit which answer. The questions will appear in order (in case there is an inadvertent typo). Correct answers will receive full credit with or without work in this section, but if you don't submit work and clearly label your answers, you won't be able to challenge any scoring decisions for making an error of any kind.

1. Find the area of the region bounded by the graphs of $f(x) = \frac{1}{1+x^2}, g(x) = \frac{1}{2}x^2$. Sketch the region. (10 points)

$$2\int_0^1 \frac{1}{1+x^2} - \frac{1}{2}x^2\,dx = 2\left[\arctan x - \frac{1}{6}x^3\right|_0^1$$

$$= 2\left[\arctan 1 - \frac{1}{6}\right] = 2\left[\pi/4 - \frac{1}{6}\right] = \frac{\pi}{8} - \frac{1}{12}$$



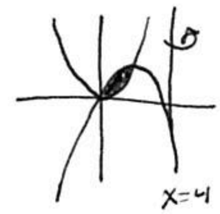
2. Find the volume of the solid of revolution generated by revolving the region bounded by the graphs $y = x^2, y = 4x - x^2$ around the line $x = 4$. You may use any method appropriate. Sketch the graph of the region and clearly indicate which method you are using. (10 points)

$$2\pi\int_0^2 (4-x)(4x - x^2 - x^2)\,dx = 2\pi\int_0^2 (4-x)(4x-2x^2)\,dx$$

$$= 2\pi\int_0^2 16x - 8x^2 - 4x^2 + 2x^3\,dx =$$

$$2\pi\int_0^2 16x - 12x^2 + 2x^3\,dx = 2\pi\left[8x^2 - 4x^3 + \frac{1}{2}x^4\right|_0^2 =$$

$$2\pi[8] = 16\pi$$



3. Find the arc length of the graph of $x = \frac{1}{3}\sqrt{y}(y-3)$ on the interval in y, [1, 4]. (8 points)

$\frac{1}{3}y^{3/2} - y^{1/2}$

$$S = \int_1^4 \sqrt{1+\left(\frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}\right)^2}\,dy =$$

$$x' = \frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{2}y^{1/2} - \frac{1}{2}y^{-1/2}$$

$$\frac{1}{2}y^{1/2} - \frac{1}{2}y^{-1/2}$$

$$= \int_1^4 \sqrt{\left(\frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\right)^2}\,dy = \int_1^4 \frac{1}{2}y^{1/2} + \frac{1}{2}y^{-1/2}\,dy =$$

$$\frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}\cdot y^{3/2} + \frac{1}{\cancel{2}}\cdot\cancel{2}\,y^{1/2}\Big|_1^4 = \frac{1}{3}(8) + 2 - \frac{1}{3}(1) - (1) = \frac{10}{3}$$

4. Find the area of the surface of revolution generated by the graph $y = \sqrt{4-x^2}$ over the interval [-1, 1] revolved around the x-axis. (10 points)

$$S = 2\pi\int_{-1}^{1} \sqrt{4-x^2}\,\sqrt{1+\left(\frac{-2}{\sqrt{4-x^2}}\right)^2}\,dx =$$

$$y' = \frac{1}{2}(4-x^2)^{-1/2}\cdot(-2x)$$

$$= -\frac{2}{\sqrt{4-x^2}}$$

$$4\pi\int_0^1 \sqrt{4-x^2}\,\sqrt{1+\frac{4}{4-x^2}}\,dx =$$

$$4\pi\int_0^1 \sqrt{4-x^2\left(1+\frac{4}{4-x^2}\right)}\,dx = 4\pi\int_0^1 \sqrt{\cancel{4}-x^2+4}\,dx = 4\pi\int_0^1 \sqrt{8-x^2}\,dx$$

$$4\pi[2.768344151] \approx 34.78803859$$

5. Find the work done in lifting a 10 ton satellite from the surface of the moon to a height of 1,000 miles. The weight given is the weight of the satellite on the moon. Assume that the radius of the moon is 1100 miles. (Gravity: $F = \frac{C}{x^2}$) (10 points)

$$F = 10 \text{ tons} = \frac{k}{1100^2}$$

$$k = 1.21\times10^7$$

$$W = \int_{1100}^{2100} 1.21\times10^7\,x^{-2}\,dx =$$

$$1.21\times10^7\left(-\frac{1}{x}\right)\Big|_{1100}^{2100} =$$

$$1.21\times10^7\left[\frac{1}{1100} - \frac{1}{2100}\right] = 5238.095\ldots$$

ton-miles

6. For the integral $\int_1^3 \frac{1}{\sqrt{x}}dx$, first calculate the number of subdivisions n that will be needed to have an Error using Simpson's Rule of less than or equal to 0.001, and then calculate the value of the integral using that method. (16 points)

$\rightarrow x^{-1/2}$

$f' = -\frac{1}{2}x^{-3/2}$

$f'' = \frac{3}{4}x^{-5/2}$

$f''' = -\frac{15}{8}x^{-7/2}$

$f^{iv} = \frac{105}{16}x^{-9/2}$

$= \frac{105}{16\sqrt{x^9}}$

max at $x=1$

$\Delta x = \frac{3-1}{6} = \frac{1}{3}$

$\{1, 4/3, 5/3, 2, 7/3, 8/3, 3\}$

$\approx \frac{1}{9}\left[\frac{1}{1} + 4\sqrt{\frac{3}{4}} + 2\sqrt{\frac{3}{5}} + 4\sqrt{\frac{1}{2}} + 2\sqrt{\frac{3}{7}} + 4\sqrt{\frac{3}{8}} + \frac{1}{\sqrt{3}}\right] \approx$

$1.4642077\ldots$

$0.001 \approx \frac{(3-1)^5\left(\frac{105}{16}\right)}{180N^4}$

$N^4 \geq \frac{32 \cdot 105/16}{180 \cdot 0.001}$

$N^4 \geq \frac{210 \cdot 1000}{180}$

$N \geq 5.844\ldots$

$N = 6$

(must be even)

7. Determine whether the series $\sum_{n=0}^{\infty} 3\left(-\frac{6}{7}\right)^n$ converge. And if so, to what. (8 points)

$|r| < 1 \rightarrow \left|-\frac{6}{7}\right| < 1$

Converges

$$\frac{3}{1-\left(-\frac{6}{7}\right)} = \frac{3}{1+\frac{6}{7}} = \frac{3}{13/7} = 3 \cdot \frac{7}{13} = \frac{21}{13}$$

8. For the series $\sum_{n=1}^{\infty} \frac{n+10}{10n+1}$ explain which test you would use and why to determine the convergence or divergence. Then determine which it does. (12 points)

diverges, using nth term test

$$\lim_{n\to\infty} \frac{n+10}{10n+1} = \frac{1}{10} \neq 0$$

9. Perform three steps of Euler's method on the differential equation $y' = e^{xy}$, starting at $(x, y) = (0,1)$. Use $\Delta x = h = 0.2$. What is your estimate for y_3? (10 points)

$m_1 = e^{(0)(1)} = e^0 = 1$

$y_1 = 1(.2) + 1 = 1.2$

$(0.2, 1.2)$

$m_2 = e^{(0.2)(1.2)} \approx 1.27124915$

$Y_2 = 1.27\ldots(0.2) + 1.2 = 1.4542\ldots$

$(0.4, 1.454249837)$

$m_3 = e^{(0.4)(1.454\ldots)} = 1.7890\ldots$

$Y_3 = (1.7890)(0.2) + 1.454\ldots = 1.81206\ldots$

$(0.6, 1.812065261) \leftarrow$

10. Determine whether the polar conics below are circles, ellipses, parabolas or hyperbolas. What is the eccentricity of each graph. (5 points each)

a. $r = \frac{3}{1+2\cos\theta}$

$e = 2$ hyperbola

b. $r = 5\cos\theta$ $e = 0$ circle

c. $r = \frac{1}{4+3\sin\theta}$

$e = \frac{3}{4}$ ellipse

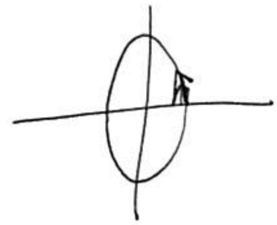
d. $r = \frac{1}{1-\cos\theta}$ $e = 1$ parabola

11. Plot the set of parametric equations $x = 3\cos t$, $y = 7\sin t$. Clearly label its orientation. Rewrite the parametric equations as a single vector-valued function. (10 points)

$\vec{r}(t) = \langle 3\cos t, 7\sin t \rangle$

or

$\vec{r}(t) = 3\cos t\, \hat{i} + 7\sin t\, \hat{j}$



12. Find the area of one petal of the graph $r = 3\cos 2\theta$. (10 points)

$3\cos 2\theta = 0$

$\cos 2\theta = 0$

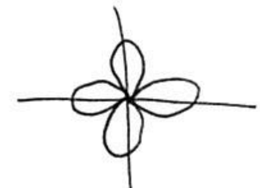
$2\theta = \pi/2, -\pi/2$

$\theta = \pi/4, -\pi/4$

$\frac{1}{2}\int_{-\pi/4}^{\pi/4} (3\cos 2\theta)^2\, d\theta =$

$\frac{1}{2}\int_{-\pi/4}^{\pi/4} 9\cos^2 2\theta\, d\theta =$

$\frac{9}{2}\int_0^{\pi/4} 1 + \cos 4\theta\, d\theta = \frac{9}{2}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi/4} = \frac{9}{2}\left(\frac{\pi}{4}\right) = \frac{9\pi}{8}$



Part 2: In this section you will record your answers on paper along with your work. After scanning, submit them to a Canvas dropbox as directed. These questions will be graded by hand.

13. Integrate using an appropriate method. (12 points each)

a. $\int x\sqrt{x-4}\,dx$

by parts

$u = x \quad dv = (x-4)^{1/2}dx$

$du = dx \quad v = \frac{2}{3}(x-4)^{3/2}$

$\frac{2}{3}x(x-4)^{3/2} - \frac{2}{3}\cdot\frac{2}{5}(x-4)^{5/2}$

$\frac{2}{3}(x)(x-4)^{3/2} - \frac{4}{15}(x-4)^{5/2} + C$

change of variable $u = \sqrt{x-4} \to u^2 = x-4$

$u^2 + 4 = x \quad 2u\,du = dx$

$\int (u^2+4)\cdot u \cdot 2u\,du = \int 2u^2(u^2+4)\,du$

$\int 2u^4 + 8u^2\,du = \frac{2}{5}u^5 + \frac{8}{3}u^3 + C$

$\frac{2}{5}(x-4)^{5/2} + \frac{8}{3}(x-4)^{3/2} + C$

b. $\int \cos^5 x \sin^2 x\,dx$

$\int \cos x \cos^4 x \sin^2 x\,dx$

$\int \cos x\,(1-\sin^2 x)^2 \sin^2 x\,dx$

$u = \sin x \quad du = \cos x\,dx$

$\int (1-u^2)^2 u^2\,du =$

$\int (1 - 2u^2 + u^4)u^2\,du =$

$\int u^2 - 2u^4 + u^6\,du$

$\frac{1}{3}u^3 - \frac{2}{5}u^5 + \frac{1}{7}u^7 + C$

$\frac{1}{3}\sin^3 x - \frac{2}{5}\sin^5 x + \frac{1}{7}\sin^7 x + C$

c. $\int \frac{x^3}{\sqrt{x^2-25}}dx$ $\quad x = 5\sec\theta \quad dx = 5\sec\theta\tan\theta\, d\theta \quad \sqrt{x^2-25} = 5\tan\theta$

$$\int \frac{125\sec^3\theta \cdot \cancel{5}\sec\theta \cancel{\tan\theta}\, d\theta}{\cancel{5\tan\theta}} = 125\int \sec^4\theta\, d\theta = 125\int \sec^2\theta(1+\tan^2\theta$$

$u = \tan\theta \quad du = \sec^2\theta\, d\theta$

$$= 125\int 1+u^2\, du = 125u + \frac{125}{3}u^3 + C$$

$$125\tan\theta + \frac{125}{3}\tan^3\theta + C = 125\sqrt{x^2-25} + \frac{125}{3}(x^2-25)^{3/2} + C$$

d. $\int \frac{x^2-1}{x^3+x}dx$

$(x)(x^2+1)$ $\quad \int \frac{-1}{x} + \frac{x}{x^2+1}\,dx = -\ln x + \frac{1}{2}\ln|x^2+1| + C$

$\frac{A}{x} + \frac{Bx+C}{x^2+1} \rightarrow Ax^2 + A + bx^2 + Cx = x^2 - 1$

$A = -1$

$A + B = 0 \rightarrow B = 1$

$C = 0$

14. Determine if the infinite series converge or diverge. Explain your reasoning, and which test you used to determine it. (10 points each)

a. $\sum_{k=1}^{\infty} \frac{2^k}{(2k+1)!}$ Converges by ratio test

$$\lim_{k\to\infty} \frac{2^{k+1}}{(2k+3)!} \cdot \frac{(2k+1)!}{2^k} = \lim_{k\to\infty} \frac{2}{(2k+3)(2k+2)} = 0$$

b. $\sum_{k=1}^{\infty} \left(\frac{2k+1}{k+1}\right)^k$ diverges by root test

$$\lim_{k\to\infty} \sqrt[k]{\left(\frac{2k+1}{k+1}\right)^k} = \lim_{k\to\infty} \frac{2k+1}{k+1} = 2 > 1$$

15. Determine if the alternating series converges conditionally, absolutely or diverges. Explain your reasoning. (8 points each)

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}+1}$$ Converges. Conditionally

$\lim_{k \to \infty} \frac{1}{\sqrt{k}+1} = 0$ Converges by ALT

limit Comparison w/ $\frac{1}{\sqrt{k}}$ diverges by p-series $p<1$

$\lim_{k \to \infty} \frac{1}{\sqrt{k}+1} \cdot \frac{\sqrt{k}}{1} = \frac{\sqrt{k}}{\sqrt{k}+1} = 1$ diverges $\to$ converges conditionally

16. Rewrite the expression $y = \frac{x^4}{(x^2-1)^3}$ as a power series. (12 points)

$-(1-x^2)^3$

$a = -x^4$

$r = x^2$

$(1-r)^{-1} = \sum_{n=0}^{\infty} ar^n$

$(1-r)^{-2} = \sum_{n=1}^{\infty} anr^{n-1}$

$2(1-r)^{-3} = \sum_{n=2}^{\infty} an(n-1)r^{n-2}$

$\frac{2}{(1-r)^3} = \sum_{n=0}^{\infty} a(n+2)(n+1)r^n$

$\sum_{n=0}^{\infty} -x^4(n+2)(n+1)(x^2)^n =$

$\sum_{n=0}^{\infty} -(n+2)(n+1)x^{2n+4}$

17. To what value (if any) does the geometric series $\sum_{n=0}^{\infty}(-4)^n 5^{-n}$ converge? If it fails to converge, explain why. (6 points)

converges $= \sum \left(\frac{-4}{5}\right)^n$ $\left|-\frac{4}{5}\right| < 1$

Sum $\frac{1}{1-\left(-\frac{4}{5}\right)} = \frac{1}{1+\frac{4}{5}} = \frac{1}{9/5} = \frac{5}{9}$

18. Find $\lim_{x \to 0} \frac{e^x - 1}{x}$ using a power series. (8 points)

$\lim_{x \to 0} \frac{\cancel{1} + x + \frac{x^2}{2} + \cdots \cancel{-1}}{x} = \lim_{x \to 0} \frac{x + \frac{x^2}{2} + \cdots}{x} = \lim_{x \to 0} 1 + \frac{x}{2} + \cdots = 1$

19. Find the solution to the initial value problem $y'(x) = \frac{\sqrt{x}}{7y}, y(0) = 2$. (6 points)

$7y\,dy = x^{1/2}dx$

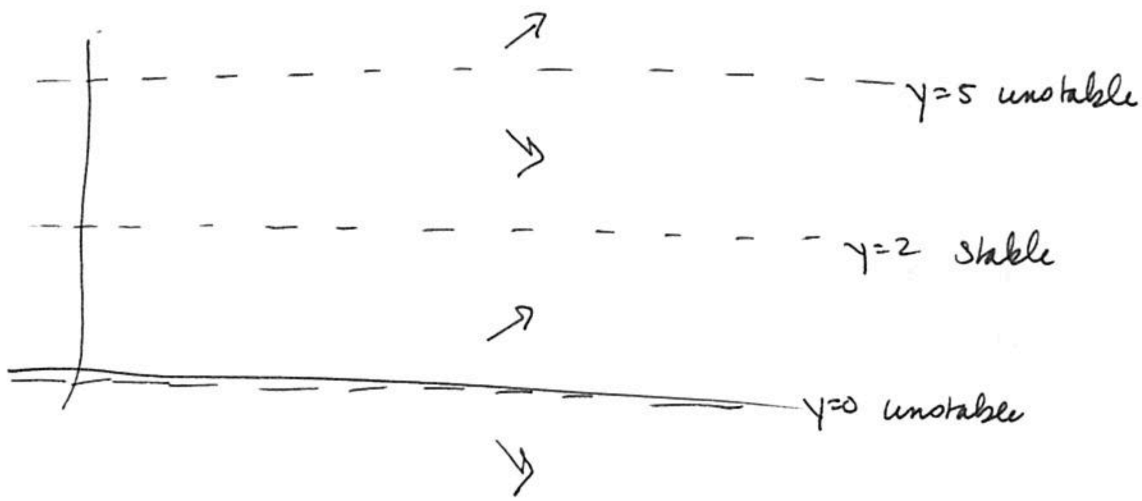
$\frac{7}{2}y^2 = \frac{2}{3}x^{3/2} + C$

$\frac{7}{2}(2)^2 = \frac{2}{3}(0)^{3/2} + C$

$14 = C$

$\frac{7}{2}y^2 = \frac{2}{3}x^{3/2} + 14$

20. Given the differential equation $\frac{dy}{dx} = y(5-y)(2-y)$. Sketch the phase plane for the equation and use that information to graph the key features of the direction field such as the equilibria (steady state solutions) and the sign of the slope in each region. Label each equilibrium as stable, unstable or semi-stable. (12 points)



21. Solve the separable differential equation $(1 + x^2)y' - 2xy = 0$. (10 points)

$(1+x^2)y' = 2xy$

$\int \frac{dy}{y} = \int \frac{2x}{1+x^2}$

$\ln y = \ln|1+x^2| + C$

$y = A(1+x^2)$

22. Find the slope of the tangent line to the graph $r = 1 + 3\cos\theta$ when $\theta = \frac{\pi}{4}$. You may use the formula $\frac{dy}{dx} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta}$. [Hint: to write the line, you may do it in rectangular coordinates, but you will have to convert the point on the polar graph to rectangular as well.] (8 points)

$r' = -3\sin\theta$

$$\frac{dy}{dx} = \frac{(-3\sin\theta)\sin\theta + (1+3\cos\theta)\cos\theta}{(-3\sin\theta)\cos\theta - (1+3\cos\theta)\sin\theta} = \frac{-3\sin^2\theta + \cos\theta + 3\cos^2\theta}{-3\sin\theta\cos\theta - \sin\theta - 3\cos\theta\sin\theta} =$$

$$\frac{-3\sin^2\theta + \cos\theta + 3\cos^2\theta}{-6\sin\theta\cos\theta - \sin\theta}$$

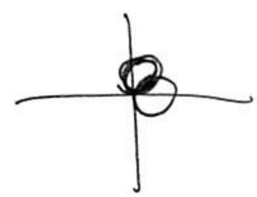
at $\pi/4$

$$= \frac{\cancel{-3\left(\tfrac{1}{2}\right)} + \left(\tfrac{1}{\sqrt{2}}\right) + \cancel{3\left(\tfrac{1}{2}\right)}}{-6\underbrace{\left(\tfrac{1}{\sqrt{2}}\cdot\tfrac{1}{\sqrt{2}}\right)}_{\frac{1}{2}} - \tfrac{1}{\sqrt{2}}} = \frac{\frac{1}{\sqrt{2}}}{-3 - \frac{1}{\sqrt{2}}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{-1}{3\sqrt{2} - 1}$$

23. Find the area of the region common to the circle $r = 2\cos\theta$, and $r = 2\sin\theta$. Sketch the region. (12 points)

$\cancel{2}\cos\theta = \cancel{2}\sin\theta$

$\theta = \pi/4$



$$\frac{1}{2}\int_0^{\pi/4} (2\sin\theta)^2\, d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} (2\cos\theta)^2\, d\theta =$$

$$\frac{1}{2}\int_0^{\pi/4} 4\sin^2\theta\, d\theta + \frac{1}{2}\int_{\pi/4}^{\pi/2} 4\cos^2\theta\, d\theta =$$

$$\int_0^{\pi/4} 1 - \cos 2\theta\, d\theta + \int_{\pi/4}^{\pi/2} 1 + \cos 2\theta\, d\theta$$

$$\theta - \frac{1}{2}\sin 2\theta\Big|_0^{\pi/4} + \left[\theta + \frac{1}{2}\sin 2\theta\right|_{\pi/4}^{\pi/2} =$$

$$\frac{\pi}{4} - \frac{1}{2}\sin(\pi/2) + \left[\underbrace{\pi/2 - \pi/4}_{\pi/4} + \cancel{\frac{1}{2}\sin\pi} - \frac{1}{2}\sin\pi/2\right] =$$

$$\frac{\pi}{2} - \frac{1}{2} - \frac{1}{2} = \frac{\pi}{2} - 1 = \frac{\pi - 2}{2}$$