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Comparison Tests: Limit and Direct Comparisons

Direct Comparison

Compare our series to another series which we already know whether it converges or diverges (by another test). If our series is a_n and the comparison series b_n converges, then we need that $a_n < b_n$ for all n. But, if the comparison series diverges, then we need $a_n > b_n$ for all n.

Consider the sequence $\frac{1}{n^2}$ vs. $\frac{1}{n^2+1}$. Which one is bigger?

$$\frac{1}{n^2+1} < \frac{1}{n^2}$$

$$\frac{1}{2} < 1, \frac{1}{5} < \frac{1}{4}, \frac{1}{10} < \frac{1}{9}, \dots$$

We know from the p-series test, that $\sum_{i=1}^{\infty} \frac{1}{i^2}$ converges, therefore, by direct comparison $\sum_{i=1}^{\infty} \frac{1}{i^2+1}$ also converges.

I can add in the denominator to make the series smaller. Or I can subtract in the numerator.

$$\sum_{n=3}^{\infty} \frac{n-2}{n^3}$$

Compare this to $\sum_{n=3}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$. This is also convergent according to the p-series test.

I need to have

$$\sum_{n=3}^{\infty} \frac{n-2}{n^3} < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{1}{27} < \frac{1}{9}, \frac{2}{64} < \frac{1}{16}, \dots$$

Therefore, since the p-series converges, and our series is always less, then we can say that our series also converges by direct comparison.

If the series we are testing diverges, then we need a series that is smaller than the one we have.
We can subtract in the denominator or we add to the numerator (relative to the simpler comparison).

$$\sum_{n=2}^{\infty} \frac{n}{n^2-1}$$

The natural comparison is to take the leading terms and try $\sum_{n=2}^{\infty} \frac{n}{n^2} = \sum_{n=2}^{\infty} \frac{1}{n}$
The harmonic series does diverge according to the p-series test, or the integral test.

Test:

$$\sum_{n=2}^{\infty}\frac{1}{n} < \sum_{n=2}^{\infty}\frac{n}{n^2-1}$$

$$\frac{1}{2} < \frac{2}{3}, \frac{1}{3} < \frac{3}{8}, \frac{1}{4} < \frac{4}{15}, \ldots$$

Since the harmonic series diverges (by the p-series test), and is smaller than our series, our series must also diverge.

We could also add to the numerator to get a bigger series.

$$\sum_{n=2}^{\infty}\frac{1}{n} < \sum_{n=2}^{\infty}\frac{n+3}{n^2}$$

We can say, that since the harmonic series diverges and is smaller than our series, then our series also diverges.

Where we run into problems is where the inequalities don't quite work:

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1}$$

The natural comparison here is $\sum_{n=2}^{\infty}\frac{1}{n^2}$, but the inequality doesn't work that we need.

$$\sum_{n=2}^{\infty}\frac{1}{n^2-1} > \sum_{n=2}^{\infty}\frac{1}{n^2}$$

We can't apply the direct comparison test here.

Sometimes the direct comparison test can be useful when working with logs, and powers of logs.

Limit Comparison

Comparing series that are going to behave similarly.
The series you are testing is a_n and the comparison series is b_n (simpler, but as similar as possible to the original, but so that you know whether it converges or diverges.)

$$\lim_{n\to\infty}\frac{a_n}{b_n} = L$$

If L is greater than 0 and finite, then the two series will converge or diverge together. If you get 0 or infinity, then the comparison doesn't work and you don't know anything. (choose a different series for comparison, or a different comparison test).

$$\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} > \sum_{n=2}^{\infty} \frac{1}{n^2}$$

Does $\sum_{n=2}^{\infty} \frac{1}{n^2-1}$ converge or diverge? The direct comparison doesn't work.

We can use this comparison for the limit comparison:

$$a_n = \frac{1}{n^2 - 1}, b_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n^2 - 1}\right)}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2}{n^2 - 1} = \lim_{n\to\infty} \frac{n^2/n^2}{n^2/n^2 - 1/n^2} = \lim_{n\to\infty} \frac{1}{1 - \frac{1}{n^2}} = 1$$

Since $\frac{1}{n^2}$ converges by the p-series, my series also converges by limit comparison test.

$$\sum_{n=0}^{\infty} \frac{3^n - 1}{4^n + 1}$$

Compare to $\sum_{n=0}^{\infty} \frac{3^n}{4^n} = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$. We know this series converges by the geometric series test since |r|<1.

$$\lim_{n\to\infty} \frac{\frac{3^n - 1}{4^n + 1}}{\frac{3^n}{4^n}} = \lim_{n\to\infty} \frac{3^n - 1}{4^n + 1}\left(\frac{4^n}{3^n}\right) = \lim_{n\to\infty} \frac{1 - 1/3^n}{1 + 1/4^n} = 1$$

By the limit comparison test, this series also converges.

The order of the ratio, $\frac{a_n}{b_n}$ or $\frac{b_n}{a_n}$ does not matter for the limit comparison test. You'll get 1/L instead of L.

Example.

$$\sum_{n=2}^{\infty} \frac{1}{\ln n}$$

What can we compare this to?

$$\ln n < n$$

$$\frac{1}{\ln n} > \frac{1}{n}$$

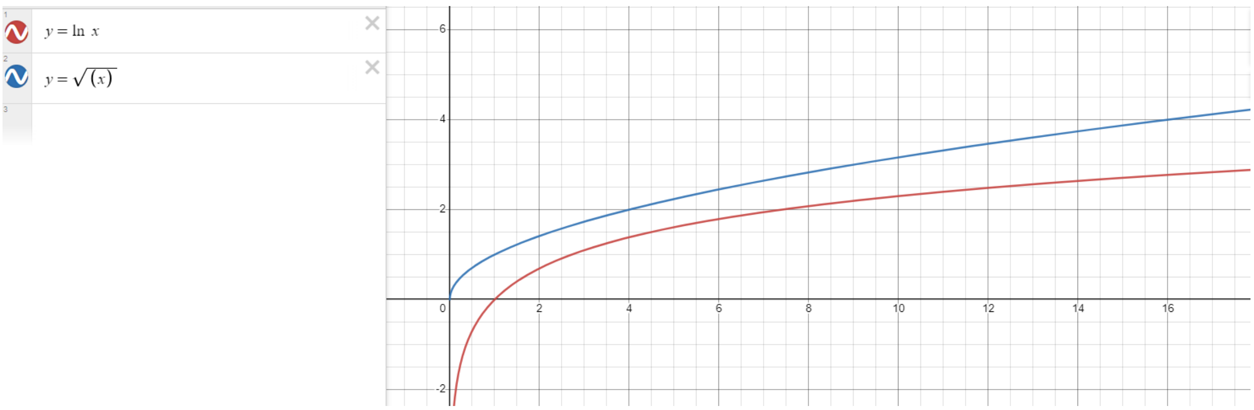
$$\sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n}$$

I already know the harmonic series diverges, and $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is bigger, therefore it also diverges.

Example.

$$\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$$

If we try comparing ln n to n, we'll run into a problem because it's too crude a comparison.



Instead of using $\ln n < n$, we're going to use $\ln n < \sqrt{n}$

$$\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2} < \sum_{n=2}^{\infty} \frac{\sqrt{n}}{n^2} = \sum_{n=2}^{\infty} \frac{1}{n^{3/2}}$$

This is a p-series which converges since p>1, by direct comparison test, the series $\sum_{n=2}^{\infty} \frac{\ln(n)}{n^2}$ also converges.

What happens if we try a limit comparison?

$$\lim_{n\to\infty} \frac{\frac{\ln(n)}{n^2}}{\frac{\sqrt{n}}{n^2}} = \lim_{n\to\infty} \frac{\ln n}{\sqrt{n}} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{2\sqrt{n}}} = \lim_{n\to\infty} \frac{2\sqrt{n}}{n} = 0$$

They behave differently enough that the limit comparison isn't valid.

Two other pieces of series that might be amenable to the comparison tests (direct comparison test in particular) is n! and n^n.

Example.

Does the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ Converge or diverge?

$$1, \frac{1}{2}, \frac{1}{6}, \frac{1}{24}, \frac{1}{120}, \frac{1}{720}, \ldots <? \; 1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \frac{1}{36}, \ldots$$

Consider $\sum_{n=5}^{\infty} \frac{1}{n!} < \sum_{n=5}^{\infty} \frac{1}{n^2}$

Since the p-series converges, so does the factorial series by direct comparison.