10/3/2023

Introduction to Series (Definition, Partial Sums, properties)
Series Tests: Geometric Series, Telescoping Series, nth term test (divergence test), Integral test, P-series

Series are sums of sequences
Infinite series: if the series, then it has a finite sum; if the series is infinite, some will have a finite sum and some will not.

The terminology is identical to improper integrals (are also sums behind the scenes): if the sum is finite then we say it converges, and any other behavior (goes it infinity, or no limit of sequence, etc.) then we call that divergent.

Series is a sum of sequence:
Sequence $\{a_n\}$
Corresponding series $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$

Partial sum adds terms of the sequence only up to a fixed value
We can create a sequence of partial sums by changing the value we stop at.

Example.

$$\{a_n\} = \frac{1}{2}n$$

$$\sum_{i=1}^{15} a_i = \frac{1}{2} + 1 + \frac{3}{2} + 2 + \frac{5}{2} + 3 + \frac{7}{2} + 4 + \frac{9}{2} + 5 + \frac{11}{2} + 6 + \frac{13}{2} + 7 + \frac{15}{2} = 60$$

Partial sums are often noted with a capital S $\{S_n\}$

$$S_1 = \sum_{i=1}^{1} a_i = \frac{1}{2}$$

$$S_2 = \sum_{i=1}^{2} a_i = \frac{1}{2} + 1 = \frac{3}{2}$$

$$S_3 = \sum_{i=1}^{3} a_i = \frac{1}{2} + 1 + \frac{3}{2} = 3$$

$$S_4 = \sum_{i=1}^{4} a_i = \frac{1}{2} + 1 + \frac{3}{2} + 2 = 5$$

The sequence of partial sums is $\{\frac{1}{2}, \frac{3}{2}, 3, 5, \ldots\}$

Another series that diverges but does not go to infinity is:

$$\sum_{i=0}^{\infty}(-1)^i$$

Look at the sequence of partial sums:

$$S_0 = \sum_{i=0}^{0}(-1)^i = 1$$

$$S_1 = \sum_{i=0}^{1}(-1)^i = 1 + (-1) = 0$$

$$S_2 = \sum_{i=0}^{2}(-1)^i = 1 + (-1) + (-1)^2 = 1$$

$$S_3 = \sum_{i=0}^{3}(-1)^i = 1 + (-1) + (-1)^2 + (-1)^3 = 0$$

$$S_4 = \sum_{i=0}^{4}(-1)^i = 1 + (-1) + (-1)^2 + (-1)^3 + (-1)^4 = 1$$

$$S_n = \{1,0,1,0,1,\ldots\}$$

This sum is divergent

What is an example of a convergent (infinite) series?

$$\sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^i$$

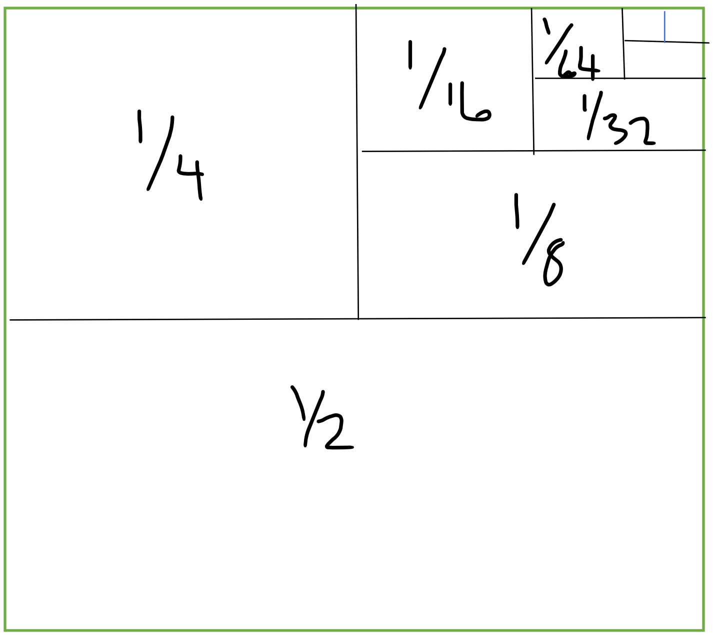
$$S_n = \{\frac{1}{2},\frac{3}{4},\frac{7}{8},\frac{15}{16},\frac{31}{32},\ldots\}$$

The limit of this partial sequence is 1:

$$S_n = \frac{2^n - 1}{2^n}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{2^n}{2^n} - \frac{1}{2^n} = 1 - 0 = 1$$

For the rest of the chapter we are going to be looking at tests for determining if the infinite series converges or diverges.



Geometric series test

Applies to series that are based on a geometric sequence $a_0 r^n$

$$\sum_{i=0}^{\infty} a_0 r^i$$

The geometric series test says that:

1) If $|r| \geq 1$, the series diverges
2) If $|r| < 1$, the series will converge
3) The infinite sum in the convergent is given by $S_\infty = \frac{a_0}{1-r}$

$$\sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^{i+1} = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^{i} = \frac{\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

Example.

$$\sum_{i=0}^{\infty} 4e^i$$

$$a_0 = 4, r = e > 1$$

This series is divergent

Example.

$$\sum_{i=0}^{\infty}\left(\frac{9}{7}\right)\left(-\frac{3}{4}\right)^{i}$$

$$a_0 = \frac{9}{7}, |r| = \frac{3}{4} < 1$$

This series will converge.
What does it converge to:

$$S_\infty = \frac{a_0}{1-r} = \frac{\left(\frac{9}{7}\right)}{1-\left(-\frac{3}{4}\right)} = \frac{\left(\frac{9}{7}\right)}{\frac{7}{4}} = \frac{9}{7}\times\frac{4}{7} = \frac{36}{49}$$

Telescoping series
Telescoping series have consecutive factors in the denominator, and when decomposed using partial fractions end up partially cancelling each other out.

$$\sum_{i=1}^{\infty}\frac{1}{(i+1)(i+2)}$$

The factors are integer spaces apart:

$$n(n+1), n(n+2), (n+1)(n+4), (2n+1)(2n+3)$$

And when decomposed they become:

$$\frac{A}{n}-\frac{A}{n+1}, \frac{B}{n}-\frac{B}{n+2}, \frac{C}{n+1}-\frac{C}{n+4}, \frac{D}{2n+1}-\frac{D}{2n+3}$$

Decompose the fraction into separate pieces.

$$\frac{A}{i+1}+\frac{B}{i+2} = \frac{\big(A(i+2)+B(i+1)\big)}{(i+1)(i+2)}$$

$$Ai + 2A + Bi + B = 1$$

$$A + B = 0$$
$$2A + B = 1$$

$$B = -A$$

$$2A + (-A) = 1$$
$$A = 1$$
$$B = -1$$

$$\sum_{i=1}^{\infty}\frac{1}{i+1}-\frac{1}{i+2}$$

$$S_1 = \frac{1}{2} - \frac{1}{3}$$

$$S_2 = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2} - \frac{1}{4}$$

$$S_3 = \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) = \frac{1}{2} - \frac{1}{5}$$

If the factors in our denominators are separated by k steps, then keep k terms at the front, and k terms at the back.
In this case, we have only one step between the factors… (i+1)+1=i+2, so we keep just the first term (the first fraction), and then last term we use the limit as that term goes to infinity

The first fraction that doesn't cancel is ½
The last term is $\lim_{i \to \infty} \frac{1}{i+2} = 0$

So the sum is $S_\infty = \frac{1}{2} - \lim_{i \to \infty} \frac{1}{i+2} = \frac{1}{2}$

$$\sum_{i=1}^{\infty} \ln\left(\frac{i}{i+1}\right) = \sum_{i=1}^{\infty} \ln(i) - \ln(i+1)$$

$$S_1 = \ln(1) - \ln 2$$

$$S_2 = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) = 0 - \ln 3$$

$$S_3 = (\ln 1 - \ln 2) + (\ln 2 - \ln 3) + (\ln 3 - \ln 4) = 0 - \ln 4$$

$$\lim_{i \to \infty} \ln(i+1) = \infty$$

Diverges

Nth term test / divergence test
If the limit of the nth term of the sequence (that the series is based on) does not converge to 0 (does have a limit of 0), then the series will diverge.

If the limit of the terms goes to 0, you know nothing.
If the limit of the terms does not go to 0, then the series diverges.

$$\sum_{i=1}^{\infty} \frac{1}{2} i$$

Test: $\lim_{i \to \infty} \frac{1}{2} i = \infty$, the sum of the series will diverge

$$\sum_{i=1}^{\infty}(-1)^i$$

Test: $\lim_{i\to\infty}(-1)^i =$ no limit

Series diverges because the limit is not 0

If the limit is 0, sometimes it will converge, and sometimes it won't…
Compare:

$$\sum_{i=1}^{\infty}\frac{1}{(i+1)(i+2)}$$

Vs.

$$\sum_{i=1}^{\infty}\ln\left(\frac{i}{i+1}\right)$$

First case: $\lim_{i\to\infty}\frac{1}{(i+1)(i+2)} = 0$, and we saw that it converged.

Second case: $\lim_{i\to\infty}\ln\left(\frac{i}{i+1}\right) = \ln\lim_{i\to\infty}\left(\frac{i}{i+1}\right) = \ln(1) = 0$, we saw that it diverged…

It is very easy to apply for divergence, but don't over apply it to claim things converge.

Integral Test:
Let $a_n = f(n)$

If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{i=1}^{\infty} a_n$ also converges
(may not be exactly the same value).

Next time we will look at the error estimate from the integral test.

$$\sum_{i=1}^{\infty}\frac{1}{i^2+1}$$

$$\int_1^{\infty}\frac{1}{x^2+1}dx = \lim_{b\to\infty}(\arctan b - \arctan 1) = \frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$$

Converges
If the integral converges, then the sum converges.