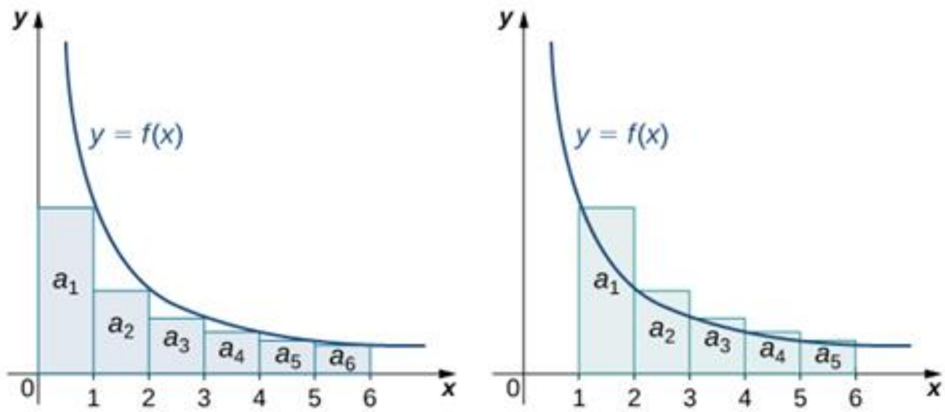
10/5/2023

Series Tests:
Continue with Integral Test, Error estimation
P-Series Test
Alternating Series Test
Comparison Tests: Direct, Limit (?)

Integral test:



Estimating the error using the Integral test.

$$\int_{N+1}^{\infty} f(x)dx \le R_N \le \int_{N}^{\infty} f(x)dx$$

$$\sum_{i=1}^{\infty} a_i = \sum_{i=1}^{N} a_i + \sum_{i=N+1}^{\infty} a_i = \sum_{i=1}^{N} a_i + R_N$$

$$R_N \le \int_{N}^{\infty} f(x)dx$$

Think of this as the maximum of the error on our finite sum.

$$\sum_{i=1}^{\infty} \frac{1}{i^2+1}$$

We established last time that this series converges.
According to the integral test:

$$\int_1^{\infty} \frac{1}{x^2+1}dx = \lim_{b\to\infty} \arctan b - \arctan 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

This establishes that the integral converges… but what does it converge to?

$$\sum_{i=1}^{10} \frac{1}{i^2+1} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10} + \frac{1}{17} + \frac{1}{26} + \frac{1}{37} + \frac{1}{50} + \frac{1}{65} + \frac{1}{82} + \frac{1}{101} = 0.9817928223 \ldots$$

Approximation for the sum using N=10 terms.

What is the maximum error on our estimate?

$$R_N = E \leq \int_N^\infty \frac{1}{x^2+1} dx = \int_{10}^\infty \frac{1}{x^2+1} dx =$$

$$\lim_{b\to\infty} \arctan b - \arctan 10 = \frac{\pi}{2} - 1.471127674 \ldots = 0.0996686525 \ldots \approx 0.1$$

Our finite sum is approximately 0.9817928223 give or take no more than 0.0996686525

The integral test is assuming that all these terms are positive.

P-series

$$\sum_{i=1}^{\infty} \frac{1}{i^p}$$

Adding up terms with changing integers in the denominator raised to a constant power…

Harmonic series: $\sum_{i=1}^{\infty} \frac{1}{i}$

Prove the p-series test by using the integral test.

There will be three cases, depending on the value of p.

1) p<1
2) p=1
3) p>1

Using the integral test: does the p-series converge or diverge?

$$a_i = f(i) = \frac{1}{i^p}, f(x) = \frac{1}{x^p} = x^{-p}$$

When p=1, this is a log-rule for integration, other values of p are the power rule.

$$\int_1^\infty x^{-p} dx = \begin{cases} \dfrac{x^{-p+1}}{1-p}, p \neq 1 \\ \ln(x), p = 1 \end{cases}$$

Case 2 (easy case): $\lim_{b\to\infty} \ln b - \ln 1 = \infty$ when $p = 1$, the integral diverges.

Case 1 (p<1): $\left(\frac{1}{1-p}\right)\left(\lim_{b\to\infty} b^{1-p}-1^{1-p}\right)$
What happens when b goes to infinity?

What if p=1/2?

$$\left(\frac{1}{1-\frac{1}{2}}\right)\left(\lim_{b\to\infty} b^{1-\frac{1}{2}}-1^{1-\frac{1}{2}}\right)=\infty$$

The power of b is positive.

Infinity raised to a positive power goes to infinity, so like the p=1 case, these cases diverge.

Case 3: p>1

$$\left(\frac{1}{1-p}\right)\left(\lim_{b\to\infty} b^{1-p}-1^{1-p}\right)$$

But suppose p=2?

$$\left(\frac{1}{1-2}\right)\left(\lim_{b\to\infty} b^{1-2}-1^{1-2}\right)$$

Any p greater than 1 will still leave the power of b negative, that means infinity is in the denominator, so the term goes to 0, and therefore the limit is finite. So the series will converge.

P-series test:
If the p in the p-series is equal to or less than 1, the series diverges, and if p is greater than 1, it converges.

$p \leq 1$: diverges
$p > 1$: converges

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}},\sum_{n=1}^{\infty}\frac{1}{n^{\frac{7}{9}}},\sum_{n=1}^{\infty}\frac{1}{n}$$

All of these diverge.

$$\sum_{n=1}^{\infty}\frac{1}{n^{1.1}},\sum_{n=1}^{\infty}\frac{1}{n^{2}},\sum_{n=1}^{\infty}\frac{1}{n^{e}}$$

These all converge

Alternating Series (5.5 in online book)

Do not confuse this test with the nth term test (divergence test)

An alternating series is a series where the signs of consecutive term flip.

$$\sum_{i=1}^{\infty}(-1)^i a_i$$

Not going to include things that change sign in an irregular fashion such as $\sin(i)$
But, watch out for expressions that act like $(-1)^n$, like $\cos(n\pi)$.

Typically $|a_{i+1}| \leq |a_i|$ (this is actually covered in the test…

If $\lim_{n\to\infty}|a_n| = 0$, then the sum of the infinite series converges. (and if it does not go to zero, the series diverges).

Example.

$$\sum_{i=1}^{\infty}(-1)^i \frac{i^2}{2^i}$$

Alternating series (with the alternating component) cannot be done with the integral test (exponential functions can't have negative bases).

Test whether the limit of the terms goes to 0:

$$\lim_{n\to\infty}\frac{n^2}{2^n} = \lim_{n\to\infty}\frac{2n}{2^n\,(\ln 2)} = \lim_{n\to\infty}\frac{2}{2^n\,(\ln 2)^2} = 0$$

This series will converge because the limit of the terms is 0, according to the alternating series test.

In the scenario where we have an alternating series, we can have situations where the series converges if it alternates, and diverges if it does not.

These series are called conditionally convergent. Series that converge regardless of the alternating part are called absolutely convergent.

$$\sum_{n=1}^{\infty}\frac{1}{n} \quad vs. \quad \sum_{n=1}^{\infty}\frac{(-1)^n}{n}$$

We know from the p-series test (and the integral test) that the non-alternating harmonic series does not converge.
But the alternating harmonic series does have a limit that goes to 0…

$$\left(1-\frac{1}{2}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\left(\frac{1}{7}-\frac{1}{8}\right)+\cdots=\left(\frac{1}{2}\right)+\left(\frac{1}{12}\right)+\left(\frac{1}{30}\right)+\left(\frac{1}{56}\right)+\cdots$$

The harmonic series is conditionally convergent.

$$\sum_{n=1}^{\infty}\frac{1}{n^2} \quad vs. \quad \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

The second one converges by the alternating series since the limit of the terms does go to zero. The absolute value of this series is the positive only series, this series converges by the p-series test. This series is absolutely convergent.

Error estimation of the alternating series test.

$$R_N \leq |a_{N+1}|$$

The maximum error on a finite sum of an alternating series compared to the infinite is bounded by the absolute value of the next term in the series. ($\pm$ error)

Example.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \approx \sum_{n=1}^{10} \frac{(-1)^n}{n} + \sum_{n=11}^{\infty} \frac{(-1)^n}{n} = \sum_{n=1}^{10} \frac{(-1)^n}{n} + R_{10} \leq \sum_{n=1}^{10} \frac{(-1)^n}{n} \pm |a_{11}|$$

$$\sum_{n=1}^{10} \frac{(-1)^n}{n} \pm |a_{11}| \approx -1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} - \frac{1}{9} + \frac{1}{10} \pm \left|\frac{1}{11}\right| \approx -0.6456 \ldots \pm 0.0909 \ldots$$

Estimating the number of terms in our finite sum to obtain an error less than a given value.

If the series is alternating, use the alternating series test to obtain the error estimate (number of terms)
If the series is not alternating, you must use the integral test.

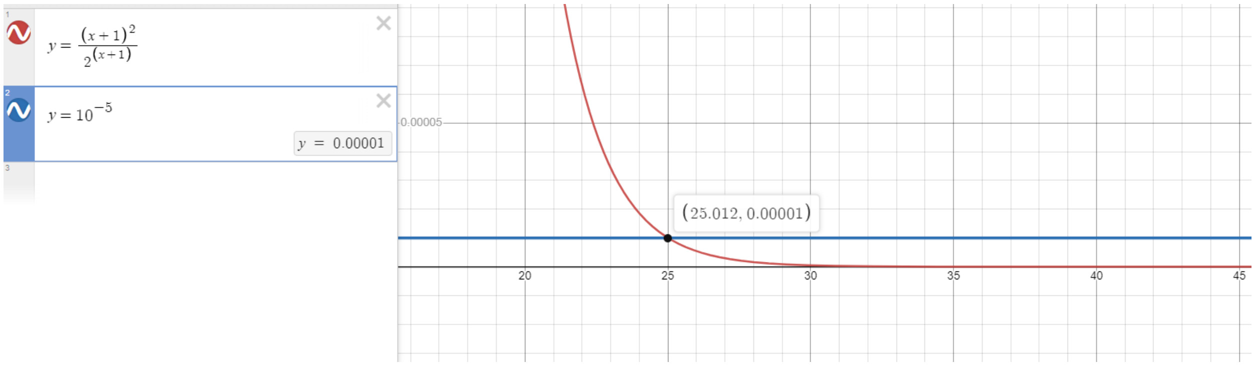
$$\sum_{i=1}^{\infty} (-1)^i \frac{i^2}{2^i}$$

Given the series, determine the number of terms needed to estimate the error on the sum to be less than 10^{-5}.

$$R_N \leq |a_{N+1}|$$

$$E \approx \frac{(N+1)^2}{2^{(N+1)}}$$

Keep in mind that the term in the error is the one after the term you stop counting at.



N=25.012…. recall that this is a threshold value, so the number of terms need is rounded up to the next whole number.
We would need N=26 terms. (because we used N+1 in the formula…)

Suppose we want to find the number of terms of $\sum_{i=1}^{\infty} \frac{1}{i^2+1}$ needed to approximate the sum with an error of 10^{-4}.

$$E \approx \int_N^\infty f(x)dx = \int_N^\infty \frac{1}{x^2+1}dx = \lim_{b\to\infty} \arctan b - \arctan N = \frac{\pi}{2} - \arctan N = 10^{-4}$$

$$\arctan N = \frac{\pi}{2} - 10^{-4} \to \tan\left(\frac{\pi}{2} - 10^{-4}\right) = N \approx 9999.999967$$

This is a threshold value, the number of terms needed is 10,000 terms.

There is no class on Tuesday for Fall Break
On Thursday we'll pick up on Comparison tests.