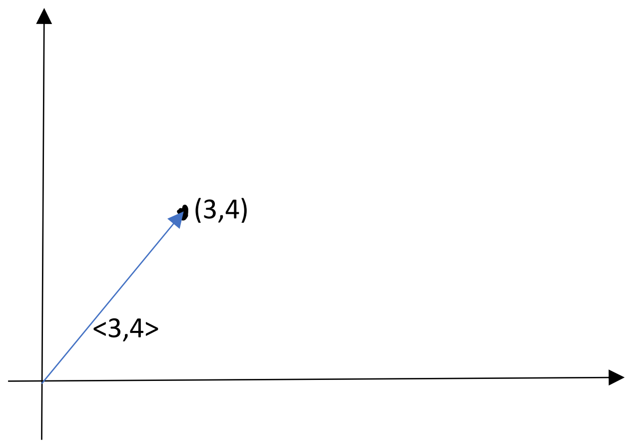
11/14/2023

Parametric Equations
Vectors

Graphing Parametric Equations, orientation
Converting to cartesian coordinates
Lines between points, circles/ellipses
Vectors in 2D, operations on vectors, unit vectors, direction and magnitude
Dot product, projections, orthogonal vectors, adding forces

Vectors
Are objects that include magnitude and direction, and also satisfy certain properties
Expressed in component form or in magnitude and direction form
When we say the wind in coming from the NE at 35 mph
When we talk about forces in physics, we say it's a force of 50N perpendicular to the surface
In component form, they look more like coordinate points… magnitude in the x direction and a magnitude in the y-direction.
Vectors by default are thought of as points in space with an arrow connecting them back to the origin.

A vector like $\langle 3,4\rangle = 3\hat{\imath} + 4\hat{\jmath}$



We can convert the component form to magnitude and direction form, and back again using polar coordinate conversions:

$$\vec{v} = \boldsymbol{v} = \langle a, b\rangle = \langle 3,4\rangle$$

Magnitude of the vector:

$$\|\vec{v}\| = \sqrt{a^2 + b^2} = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

Direction is given by:

$$\tan^{-1}\left(\frac{b}{a}\right) \, possibly + \pi = \theta$$

Theta is the angle relative to the positive x-axis, recall that inverse tangent only gives angles in the first and fourth quadrants, so if the endpoint of the vector is in the 2nd or third quadrants you'll need to add that pi.

$$\tan^{-1}\left(\frac{4}{3}\right) = 0.927 \ldots radians$$

Likewise we can convert direction and magnitude into component form:

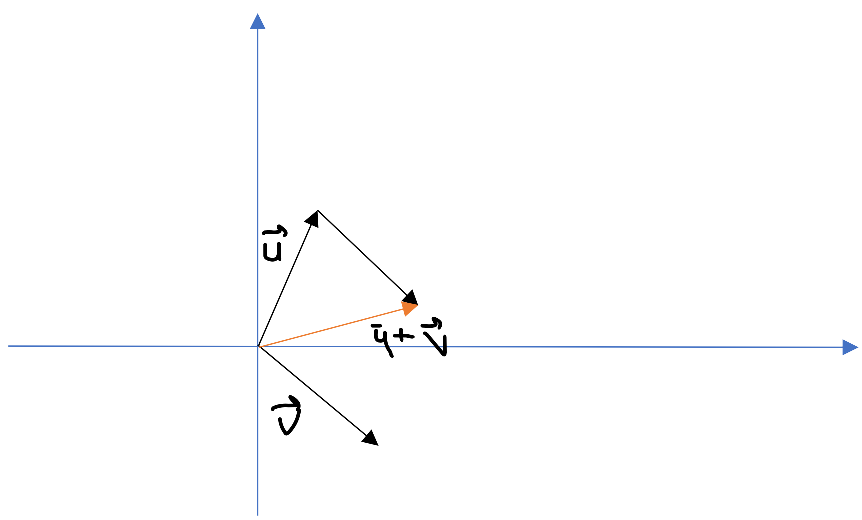
$$a = \|\vec{v}\| \cos\theta \, , b = \|\vec{v}\| \sin\theta$$

Suppose we have a force of 15N acting at an angle of 30-degrees with the horizontal, find the component form of the vector:

$$\langle 15\cos\left(\frac{\pi}{6}\right), 15\sin\left(\frac{\pi}{6}\right)\rangle = \langle \frac{15\sqrt{3}}{2}, \frac{15}{2}\rangle$$

Operations on vectors:
Vectors add component by component

$$\langle 1,3\rangle + \langle 2,-2\rangle = \langle 3,1\rangle$$



$\vec{u} - \vec{v}$ is the diagonal connector between the two arrows of the vectors if they start at the same point.
$\vec{u} + \vec{v}$ is the diagonal connector between the starting and the end of the second arrow if that second vector starts where the first one ended (shown in the drawing).

Scale vectors: all components get multiplied by the multiplier

$$k\langle a, b\rangle = \langle ka, kb\rangle$$

$$2\langle 3,4\rangle = \langle 6,8\rangle$$

Unit vectors:
To obtain a unit vector from another vector, we divide the vector by its magnitude. This vector can then be used to specify a direction.

$$\hat{v} = \frac{\vec{v}}{||\vec{v}||} = \frac{\langle 3,4 \rangle}{5} = \langle \frac{3}{5}, \frac{4}{5} \rangle$$

This operation preserves the direction, but just rescales the length. The unit vector always has a length of 1.

$$\langle \cos\theta, \sin\theta \rangle = \langle \frac{3}{5}, \frac{4}{5} \rangle$$

Dot product:

$$\vec{u} = \langle a, b \rangle, \vec{v} = \langle c, d \rangle$$
$$\vec{u} \cdot \vec{v} = ac + bd$$

Sometimes this is called an inner product, scalar product (because the answer is a scalar (number)).

Find the angle between two vectors:

$$\theta = \cos^{-1}\left(\frac{\vec{u} \cdot \vec{v}}{||\vec{u}|| ||\vec{v}||}\right)$$

$$\langle 1,3 \rangle \cdot \langle 2, -2 \rangle = (1)(2) + (3)(-2) = 2 - 6 = -4$$

$$\theta = \cos^{-1}\left(-\frac{4}{\sqrt{10}\sqrt{8}}\right) = 2.034 \ldots radians\ or\ 116.565 \ldots degrees$$

If the dot product is positive, then the angle between the vectors is acute. If the dot product is negative, the angle is obtuse. If the dot product is 0, then the vectors are perpendicular (the angle is a right angle).

Perpendicular = orthogonal

Want a vector perpendicular to $\langle 1,3 \rangle$: $\langle 1,3 \rangle \cdot \langle a, b \rangle = 0$

$$a + 3b = 0$$
$$a = -3b$$

$$\langle -3,1 \rangle$$

In two dimensions: switch the coordinates and change one sign.

Projection of vectors:

$$proj_{\vec{u}}(\vec{v}) = \left(\frac{\vec{u} \cdot \vec{v}}{||\vec{u}||^2}\right)\vec{u}$$

Find the projection of $\langle 2, -2 \rangle$ onto $\langle 1,3 \rangle$

$$proj_{\vec{u}}(\vec{v}) = -\frac{4}{10}\langle 1,3\rangle = \langle -\frac{2}{5}, -\frac{6}{5}\rangle$$

Adding forces: forces in magnitude and direction form, but to add them they need to be in component form, to give the resulting force, they need to go back into magnitude and direction form.

Two forces are being applied to an object. One had a force of 10N at an angle of 45 degrees with the horizontal. And a second force has a magnitude of 20N at an angle of 30 degrees with the horizontal. What is the resulting magnitude and direction of the total force?

$$\vec{F}_1 = \langle 10\cos\left(\frac{\pi}{4}\right), 10\sin\left(\frac{\pi}{4}\right)\rangle = \langle \frac{10}{\sqrt{2}}, \frac{10}{\sqrt{2}}\rangle$$

$$\vec{F}_2 = \langle 20\cos\left(\frac{\pi}{6}\right), 20\sin\left(\frac{\pi}{6}\right)\rangle = \langle 10\sqrt{3}, 10\rangle$$

$$\vec{F}_T = \langle \frac{10}{\sqrt{2}} + 10\sqrt{3}, \frac{10}{\sqrt{2}} + 10\rangle$$

It may be worth converting to decimals at this point.

$$\|\vec{F}_T\| = \sqrt{\left(\frac{10}{\sqrt{2}} + 10\sqrt{3}\right)^2 + \left(\frac{10}{\sqrt{2}} + 10\right)^2} = \sqrt{(24.39\ldots)^2 + (17.07\ldots)^2} = \sqrt{886.37\ldots} \approx 29.77\ldots$$

$$\theta = \tan^{-1}\left(\frac{\frac{10}{\sqrt{2}} + 10}{\frac{10}{\sqrt{2}} + 10\sqrt{3}}\right) = 34.98°$$

Parametric equations

They are a way of representing a function of x and y in terms of a third variable. $x(t), y(t)$, and a point on the curve is defined by a pair of values for x and y at the same value of t.

It can allow us to represent non-functions (in and x and y) as functions of t with two separate equations.

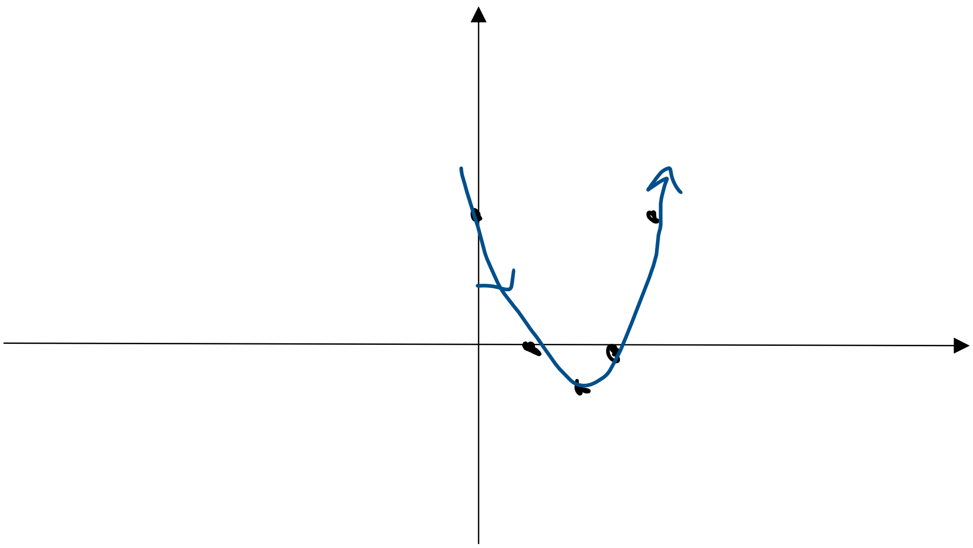
$$x(t) = t + 2$$
$$y(t) = t^2 - 1$$

Plot parametric curves:

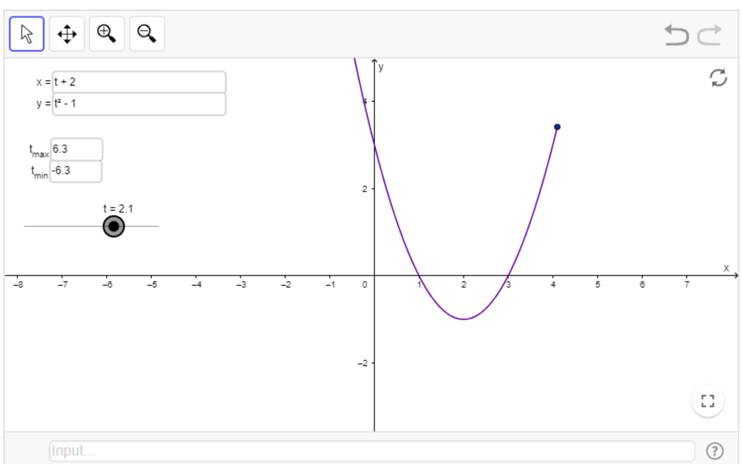
Pick values of t, plug into to both x and y, and then plot the points and connect in increasing t

t	x	y
-2	0	3
-1	1	0
0	2	-1
1	3	0

2	4	3



https://www.geogebra.org/m/cAsHbXEU



To convert back to cartesian coordinates (just x and y), solve for t in one equation, and plug into the other.

$$x = t + 2$$
$$x - 2 = t$$
$$y = t^2 - 1 = (x - 2)^2 - 1$$

$$y = x^2 - 4x + 4 - 1 = x^2 - 4x + 3$$

If you have the cartesian form, if it’s in explicit form, you can always convert to a parametric form by replacing x with t, and then having the system

$$x(t) = t$$

$$y(t) = f(t)$$

There is more than one way to make an equation parametric. Can satisfy other time-based properties, it will follow the same path, but do so differently.

We go directly to the parametric form of a line without going through $y = mx + b$.

Find the parametric equation for a line between (-1,4) and (2,8).

$$x = \Delta x(t) + x_1$$
$$y = \Delta y(t) + y_1$$

When t=0, you will be at the first point, and when t=1, you'll be at the second point.

$$\Delta x = 2 - (-1) = 3$$
$$\Delta y = 8 - 4 = 4$$

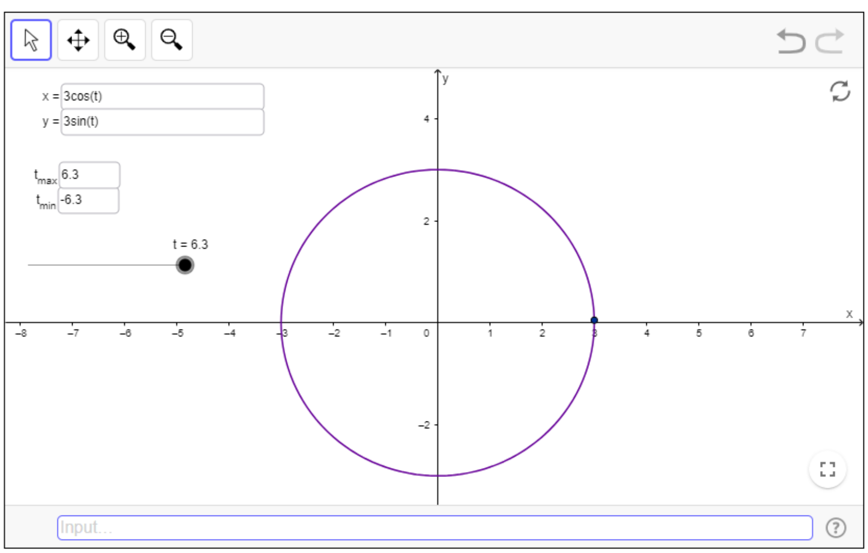
$$x = 3t - 1$$
$$y = 4t + 4$$

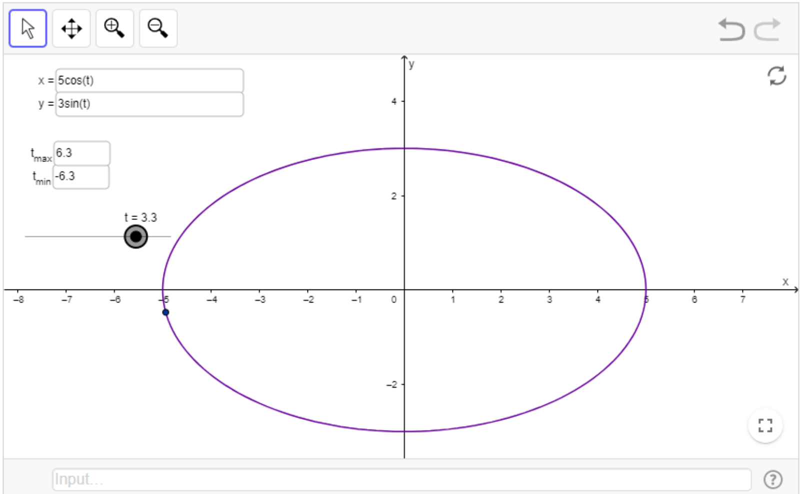
Some special functions that parametrized in particular ways:

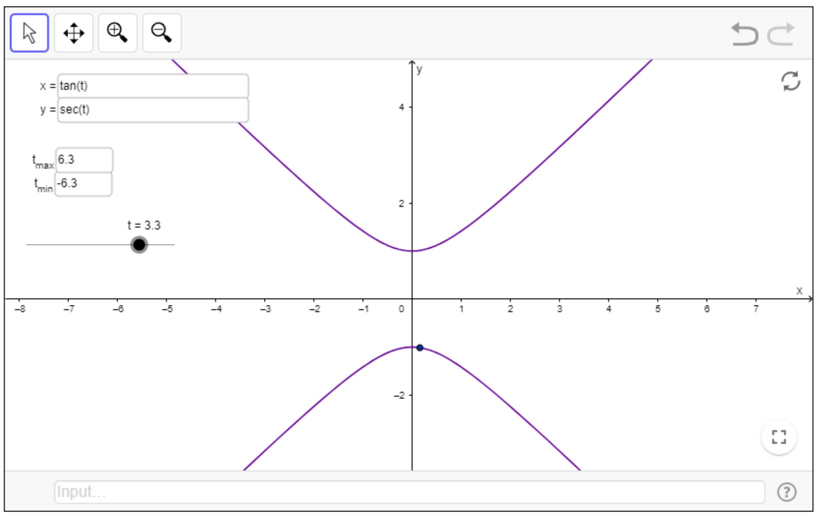
Circles: $x = a\cos t\,, y = a\sin t$

Ellipses: $x = a\cos t\,, y = b\sin t$

Hyperbolas: $x = a\tan t\,, y = b\sec t$







$$\cos^2 t + \sin^2 t = 1$$

$$1 + \tan^2 t = \sec^2 t$$

The way that this relates to vectors is that we can think of any point on a parametric curve as the endpoint of a vector. And so we can represent the entire curve as a vector:

$$\vec{r}(t) = \langle t + 2, t^2 - 1 \rangle$$

$$\vec{r}(t) = \langle 3 \cos t \,, 3 \sin t \rangle$$