11/2/2023

Go over Exam #2
Differential Equations Basics
Slope/Direction Fields
Numerical Methods

Differential Equations are equations that involve derivatives.

$$\frac{dy}{dx} = y$$

The function y is dependent on x. And its derivative is equal to the original function.

The solution to this differential equation is $y = Ae^x$

$$\frac{dy}{dx} = \frac{d}{dx}[Ae^x] = A\frac{d}{dx}[e^x] = Ae^x = y$$

$$\frac{dy}{dx} = x^2 + 1$$

$$\int \frac{dy}{dx} dx = \int x^2 + 1\, dx$$

$$y = \frac{1}{3}x^3 + x + C$$

Differential equations by themselves will produce a family of functions that differ only by a constant.

To specify the solution to just a single equation, we need to specify a set of points (maybe just a single point) that the solution passes through. The exact number of points is going to depend on whether the equation is linear and the number of derivatives (highest derivative) in the equation.

When such points are specified, the problem is collectively referred to as an initial value problem.

$$\frac{dy}{dx} = x^2 + 1, y(0) = 4$$

$$4 = \frac{1}{3}(0)^2 + (0) + C$$

$$C = 4$$

The particular solution that solves the initial value problem is $y = \frac{1}{3}x^3 + x + 4$

Verify that a function is a solution to the differential equation:
Verify that $y = \frac{1}{3}x^3 + x + 4$ is a solution to the differential equation $\frac{dy}{dx} = x^2 + 1$.

$$\frac{dy}{dx} = \frac{d}{dx}\left[\frac{1}{3}x^3 + x + 4\right] = x^2 + 1$$

Classifying differential equations

Linearity – the relationships between y and its derivatives – okay to multiply y and its derivatives by constants or by functions of x (the independent variable), but not functions of y, or other derivatives of y.
Degree – the power that any function of y or its derivatives are raised to
Order – the highest derivative in the equation
Ordinary or Partial – ordinary have only one independent variable, while partial differential use partial derivative

Linearity

Linear:

$$\frac{dy}{dx} = ky + 5$$

$$\frac{dy}{dx} = \frac{d^2y}{dx^2} + y \ \rightarrow y' = y'' + y$$

$$\frac{d^3y}{dx^3} = e^x \frac{dy}{dx} + y$$

Nonlinear:

$$\left(\frac{dy}{dx}\right)^2 = y$$

$$y'y = y''$$

$$\frac{dy}{dx} = \sin{(y)}$$

The non-linearity vs. linearity depends on what y is doing, not the independent variable.

Order:
Depends the highest derivative in the equation.
First:

$$\frac{dy}{dx} = \sin{(y)}$$

$$\left(\frac{dy}{dx}\right)^2 = y$$

$$\frac{dy}{dx} = ky + 5$$

Second:

$$y'y = y''$$

$$y' = y'' + y$$

Third:

$$\frac{d^3y}{dx^3} = e^x \frac{dy}{dx} + y \rightarrow y''' = e^x y' + y$$

Ordinary: all the examples we've looked at so far are ordinary differential equations. The derivatives are given in terms of d, or $y', y'', etc.$

Partial derivatives use $\partial$ instead of d, and use subscripts for the derivatives $u_x = \frac{\partial u}{\partial x}, u_{xx} = \frac{\partial^2 u}{\partial x^2}, u_{xy} = \frac{\partial^2 u}{\partial y \partial x}$.

Ordinary:

$$y'' + 6y' - 11y = e^x$$

Partial:

$$u_x = u_y$$

$$u_{xx} - u_{yy} = u$$

$$u_x u_y = u_{xx}$$
$$u_x = \cos(u)$$

Example.
Determine whether the differential equation is 1) ordinary or partial, 2) linear or nonlinear, 3) its order.

$$\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} - e^y = \arctan x$$

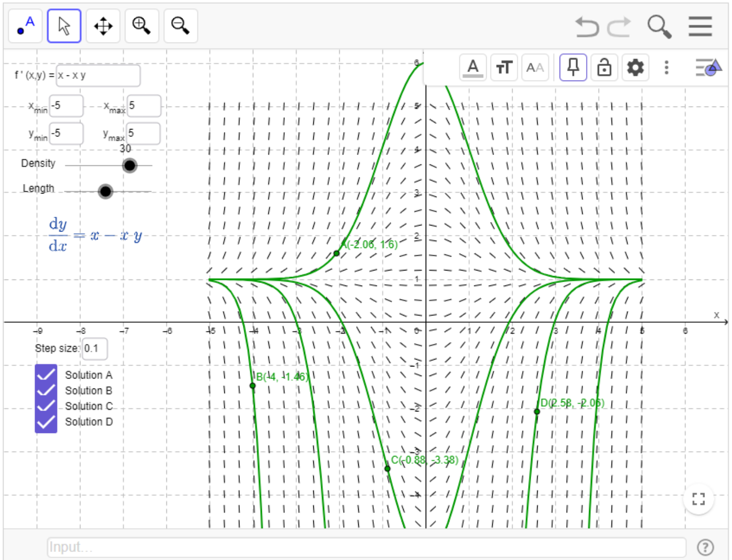
1) Ordinary, 2) nonlinear, 3) second order

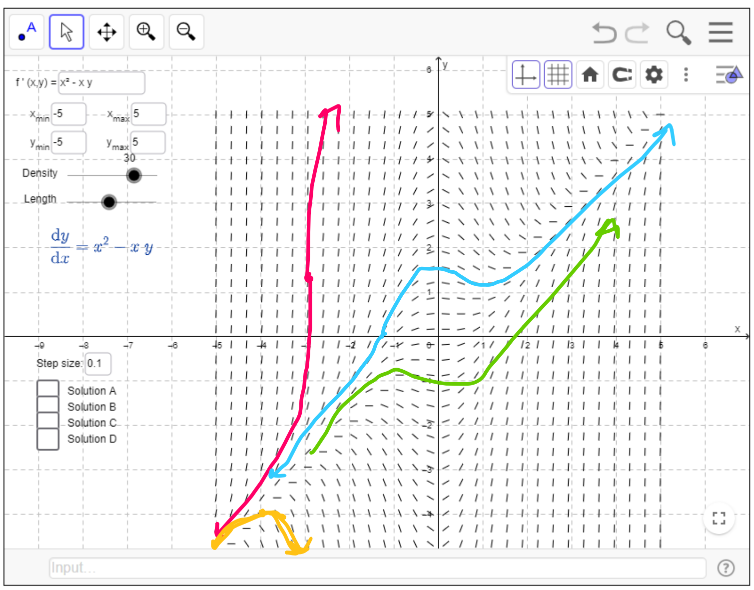
$$u_{xxx} - u_{xxy} + u = x^2 y^2$$

1) Partial, 2) linear, 3) third order

Direction Fields/Slope Fields (first derivative is the slope)

https://www.geogebra.org/m/W7dAdgqc (an example)




Euler's Method

It's an iterative process where you estimate the linear approximation of the solution from your starting point. You use that approximation to move your starting point, and re-estimate to make the next step. Do this repeatedly until you get to the x value where you are hoping to approximate the y value.

$$y_{n+1} = m_n \Delta x + y_n$$

$$m_n = \frac{dy}{dx}(x_n, y_n)$$

$\frac{dy}{dx} = x^2 - xy$, $y(0) = 3$, estimate the value of $y(2)$ using 4 steps.

$$\Delta x = h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$$

$$x_0 = 0, y_0 = 3$$

$$m_0 = 0^2 - 0(3) = 0$$

$$y_1 = 0\left(\frac{1}{2}\right) + 3 = 3$$

$$x_1 = x_0 + \Delta x = 0 + \frac{1}{2} = \frac{1}{2}$$

$$x_1 = \frac{1}{2}, y_1 = 3$$

$$m_1 = \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)(3) = \frac{1}{4} - \frac{3}{2} = -\frac{5}{4}$$

$$y_2 = -\frac{5}{4}\left(\frac{1}{2}\right) + 3 = \frac{19}{8}$$

$$x_2 = \frac{1}{2} + \frac{1}{2} = 1$$

$$m_2 = (1)^2 - 1\left(\frac{19}{8}\right) = -\frac{11}{8}$$

$$y_3 = -\frac{11}{8}\left(\frac{1}{2}\right) + \frac{19}{8} = \frac{27}{16}$$

$$x_3 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$m_3 = \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)\left(\frac{27}{16}\right) = -\frac{9}{32}$$

$$y_4 = -\frac{9}{32}\left(\frac{1}{2}\right) + \frac{27}{16} = \frac{99}{64} = 1.546875$$

Estimate for $y(2) = \frac{99}{64}$

Euler’s can run into problems near asymptotes (where the differential equation may not be defined). Be vary of vertical tangents.