11/30/2023

Conic Sections
Review in Rectangular Coordinates
Conics in Polar Coordinates

Circle

$$(x-h)^2+(y-h)^2=r^2$$

The radius is r. The center is (h,k).

So a circle centered at the origin is just $x^2+y^2=r^2$.

General form of a circle:

$$x^2+y^2+Ax+By+C=0$$

To find the center and radius one would need to complete the square for both x and y.
This will still be a circle if the coefficients of x^2 and y^2 are both the same, and both same sign.

Example.

$$x^2+y^2-2x+4y=8$$

$$(x^2-2x)+(y^2+4y)=8$$

$$(x^2-2x+1)+(y^2+4y+4)=8+1+4$$

$$(x-1)^2+(y+2)^2=13$$

$$center: (1,-2), radius: \sqrt{13}$$

In polar coordinates:

$$r=3$$

Centered at the origin

$$r=3\cos\theta\,, r=2\sin\theta$$

Circles, but shifted off the origin (pass through the origin, but not centered at the origin)
The coefficient out front is the diameter, and the center is half that, and the radius is half that.

$$r=2\sin\theta$$

Center is $(0,1)$, and the radius is 1.

Eccentricity: for a circle, the eccentricity is 0.

Ellipse

In rectangular coordinates:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

$$b^2(x-h)^2 + a^2(y-k)^2 = a^2b^2$$

(The area of an ellipse is $A = \pi ab$)

Typically, a is the longest side and a>b, and $a^2 = b^2 + c^2$, c is the distance to the focus from the center. (h, k) is the center, a is the distance from the center to the furthest point on the ellipse (semi-major axis), and b is the distance from the center to the closest point on the ellipse (semi-minor axis), and c is the distance from the center to the focus.

The distance from focus 1 to the ellipse and then back to focus 2 is always the same distance.

General form:

$$Ax^2 + By^2 + Cx + Dy + E = 0$$

Example.

$$9x^2 + 4y^2 - 36x + 24y + 36 = 0$$

$$(9x^2 - 36x) + (4y^2 + 24y) = -36$$
$$9(x^2 - 4x) + 4(y^2 + 6y) = -36$$
$$9(x^2 - 4x + 4) + 4(y^2 + 6y + 9) = -36 + 36 + 36$$

$$9(x-2)^2 + 4(y+3)^2 = 36$$

$$\frac{9(x-2)^2}{36} + \frac{4(y+3)^2}{36} = \frac{36}{36}$$

$$\frac{(x-2)^2}{4} + \frac{(y+3)^2}{9} = 1$$

Center is at $(2, -3)$, and $a = 3, b = 2, c = \sqrt{5}$
Eccentricity of any conic (other than the circle) as $\frac{c}{a} = e$
In this ellipse: $e = \frac{\sqrt{5}}{3} < 1$
All ellipses have an eccentricity $0 < e < 1$.

Parabola
Eccentricity =1

In rectangular coordinates:

$$y - k = \frac{1}{4p}(x-h)^2$$

$$y = \frac{1}{4p}(x-h)^2 + k$$

Center (Vertex) is the point (h, k). The distance to the focus (or the directrix) from vertex is p.

General form has only one square

$$Ax^2 + Bx + Cy + D = 0$$
$$Ay^2 + Bx + Cy + D = 0$$

Hyperbola

Standard form:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

The sign determines the orientation. There is no sign relationship between a and b directly. $a^2 + b^2 = c^2$, where c is the distance to the focus. And (h, k) is the center. $c > a, b$

Eccentricity $e = \frac{c}{a} > 1$

The typical way of plotting a hyperbola (after plotting the center, the vertices (the a distance from center), is to plot the asymptotes that the hyperbola approaches. A box is constructed around the center, the sides are a distance on either side of the center, and b units and down from the center. Then draw lines through the corners. The slopes of these lines are either $\pm\frac{a}{b}$ or $\pm\frac{b}{a}$ depending on the orientation of the hyperbola.

$$Ax^2 - By^2 + Cx + Dy + E = 0$$

(A and B are different signs, it could be $-Ax^2 + By^2$ ...)
If you are completing the square, be extra careful of the negative, because that will effect the signs of the coefficients and the balancing constant.

Conics in Polar

Review: circles are weird: $r = 3, r = 3\cos\theta$

Other conics will have the general form:

$$r = \frac{A}{a \pm b\cos\theta}, r = \frac{A}{a \pm b\sin\theta}$$

Want to make a =1 to find the eccentricity

$$r = \frac{A}{1 \pm e\cos\theta}$$

e is the eccentricity.
For example:

$$r = \frac{3}{1 + 2\sin\theta}$$

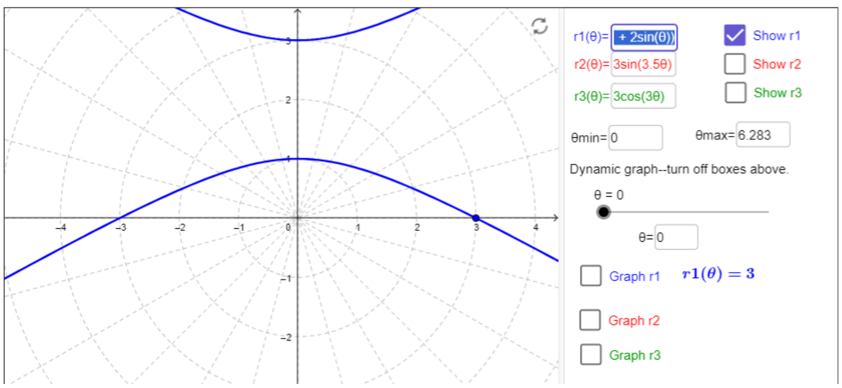
$$r = \frac{1}{4 - 5\cos\theta}$$

$$r = \frac{2}{3 + 3\sin\theta}$$

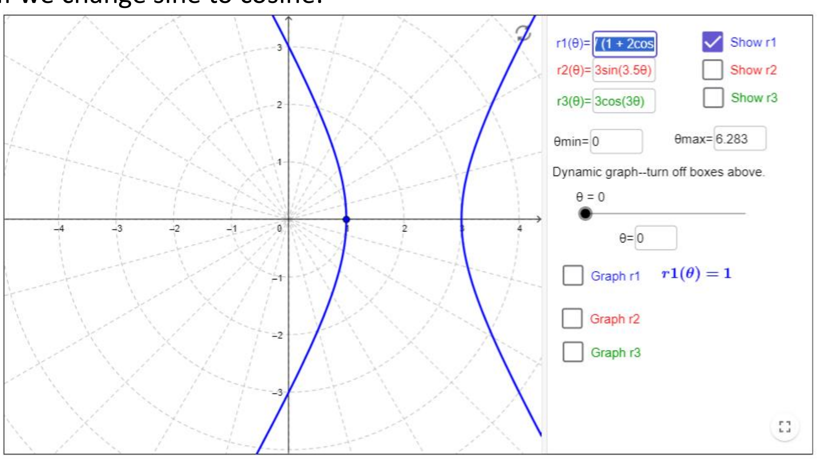
$$r = 8\cos\theta$$

What type of conic is each equation?

In the first equation, the constant is already 1 in the denominator, so $e = 2$. This is a hyperbola.



If we change sine to cosine:



In the second equation, if I divide the 4 out of the denominator:

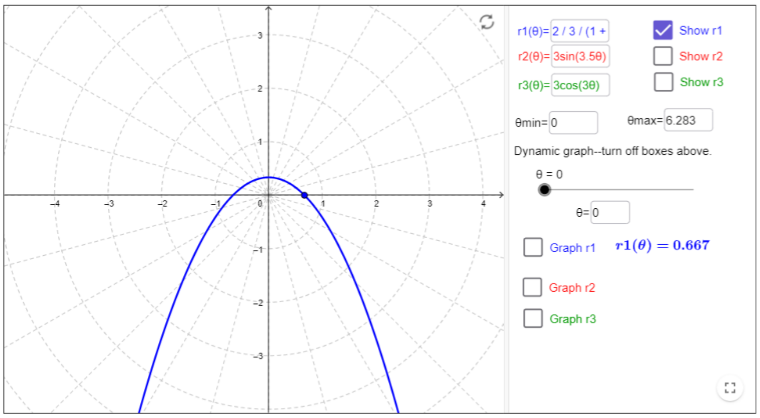
$$r = \frac{\frac{1}{4}}{1 - \frac{5}{4}\cos\theta}$$

This is also a hyperbola, because $e = \frac{5}{4}$

In the third equation:

$$r = \frac{\left(\frac{2}{3}\right)}{1 + \sin\theta}$$

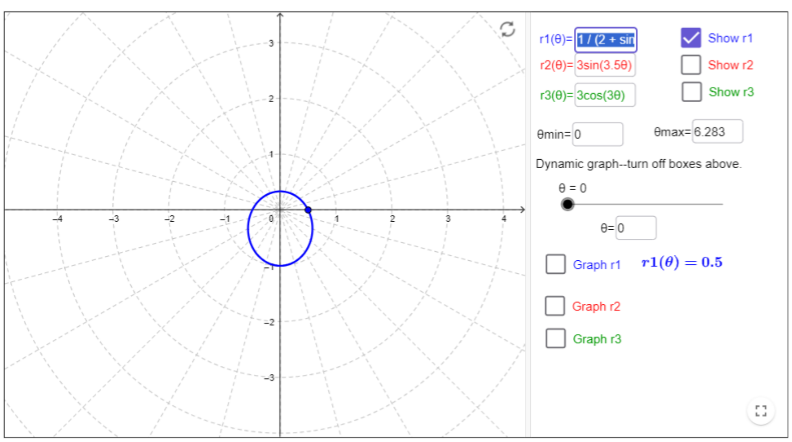
$e = 1$ which is a parabola.



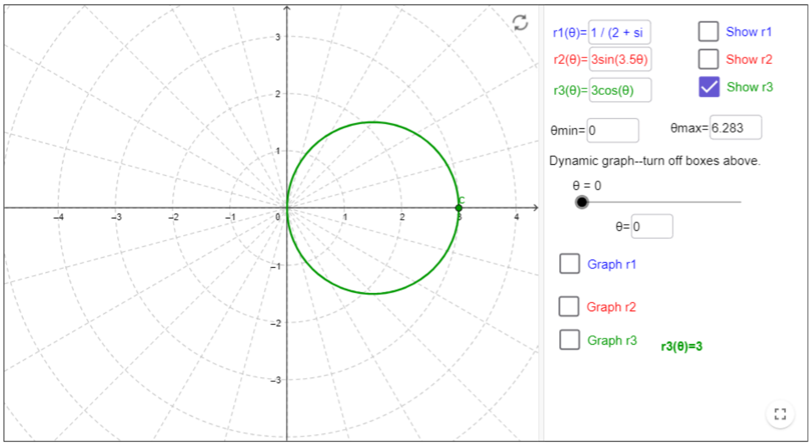
The 4th one is a circle.

$$r = \frac{1}{2 + \sin\theta} = \frac{\left(\frac{1}{2}\right)}{1 + \frac{1}{2}\sin\theta}$$

$e = \frac{1}{2}$ this is an ellipse.



The circle:



r1(θ)= 1 / (2 + si
Show r1
r2(θ)= 3sin(3.5θ)
Show r2
r3(θ)= 3cos(θ)
Show r3
θmin= 0
θmax= 6.283
Dynamic graph--turn off boxes above.
θ = 0
θ= 0
Graph r1
Graph r2
Graph r3
r3(θ)=3