8/22/2023

Introduction to the course
Area between curves

Area between curves (2.1)

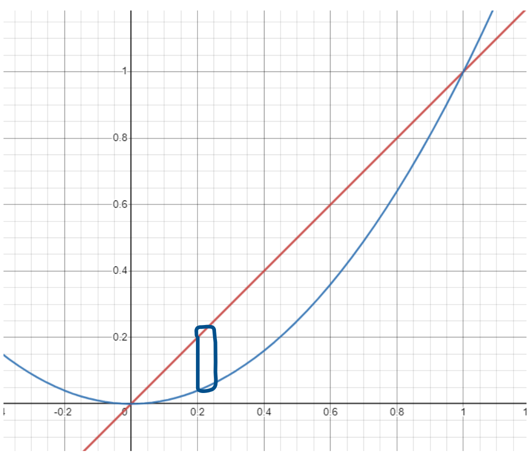
In Calc I, we covered finding the area under a curve (between the curve and the x-axis).

Method #1: look at how the heights of the rectangles change when we calculate the heights between two curves vs. between the x-axis and a curve.

Method #2: generalizing the original integral using a different function than the x-axis.

Method #3: calculating the area under the top curve, and removing the area under the bottom curve.

https://www.desmos.com/calculator



Find the area between the curves $f(x) = x, g(x) = x^2$. (on the interval [0,1] where the intersections are)

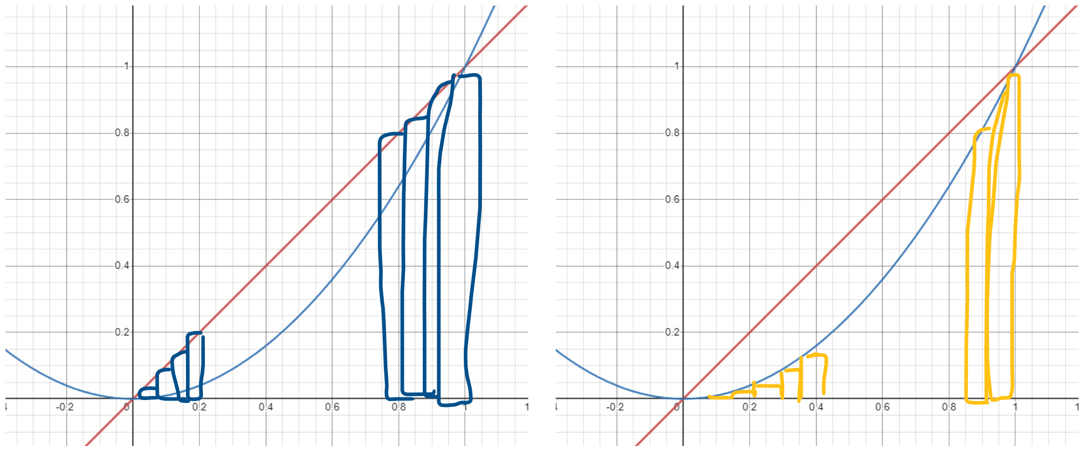
$$A = \sum_{i=1}^{n} (height)(width) = \sum_{i=1}^{n} [f(x_i) - g(x_i)]\Delta x = \sum_{i=1}^{n} f(x_i)\Delta x - \sum_{i=1}^{n} g(x_i)\Delta x$$

The limit can be applied to both summations resulting in

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$$

In the original set-up, the “second function” was the x-axis, which is $g(x) = 0$.

$$A = \int_a^b [f(x) - g(x)]dx = \int_a^b [f(x) - 0]dx = \int_a^b f(x)dx$$



What is the area between curves? It's the area below the top curve, but then take away the part I don't need under the bottom curve, and the result will between what is left between them.

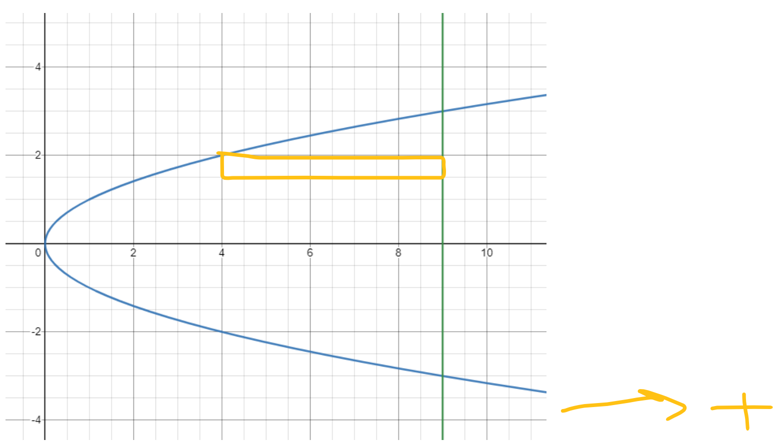
$$A = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b [f(x) - g(x)]dx$$

In general, f(x) is the top function and g(x) is the bottom function.

Example.
Find the area between the curves $f(x) = x, g(x) = x^2$.

$$A = \int_0^1 x - x^2 dx = \frac{1}{2}x^2 - \frac{1}{3}x^3 \Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

Example.
Find the area between the curves $x = y^2$ and $x = 9$.



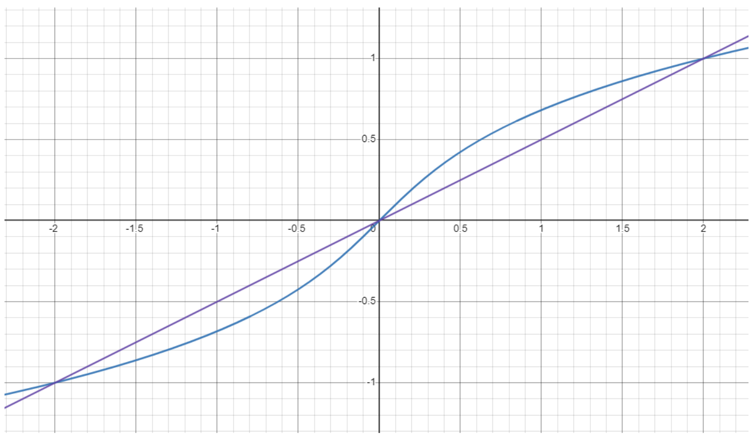
When working with functions of y (instead of functions of x), our rectangles are horizontally oriented, and not vertically. The rightmost function becomes the "top" function because it has a larger value in x. And the leftmost function becomes the "bottom" function because it has a smaller value in x.

Intersections at $y^2 = 9 \rightarrow y = \pm 3$

$$A = \int_{c}^{d} \left(f_{right}(y) - g_{left}(y)\right) dy = \int_{-3}^{3} 9 - y^2 dy = 2\int_{0}^{3} 9 - y^2 dy =$$

$$2[9y - \frac{1}{3}y^3\Big|_0^3 = 2[27 - 9] = 36$$

Example.
Find the area between $x = y + y^3$ and $2y = x$



Split the area calculation where the functions change orientations: here, at 0, because the line is the right function on the right side of the graph with the cubic on the left; but on the left side of the graph, the line is the left function and the cubic is the right function.

Set the equations equal to each other to get the intersections:

$$y + y^3 = 2y$$
$$y^3 - y = 0$$
$$y(y^2 - 1) = 0$$
$$y(y - 1)(y + 1) = 0$$

$$y = 0, -1, 1$$

$$A = \int_{-1}^{0} [(y + y^3) - (2y)]dy + \int_{0}^{1} [(2y) - (y + y^3)]dy =$$

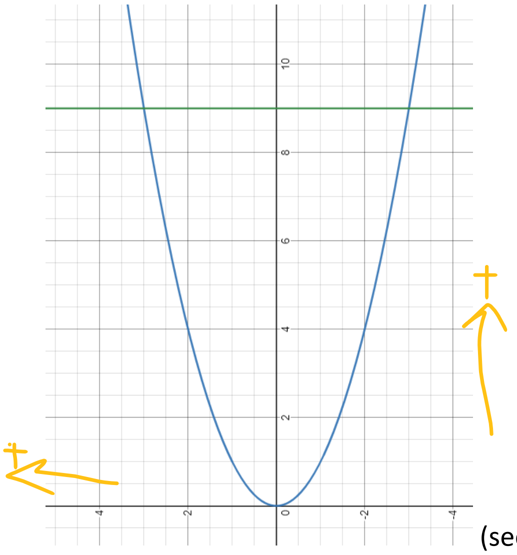
$$\int_{0}^{1} [(2y) - (y + y^3)]dy = \int_{0}^{1} [y - y^3]dy = \left[\frac{1}{2}y^2 - \frac{1}{4}y^4\right]_0^1 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$$\int_{-1}^{0} [(y + y^3) - (2y)]dy = \frac{1}{4}$$

$$A = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

If you want to escape the horizontal orientation, there are some tricks you can use:

One option is to keep everything the same, but rotate your graph.



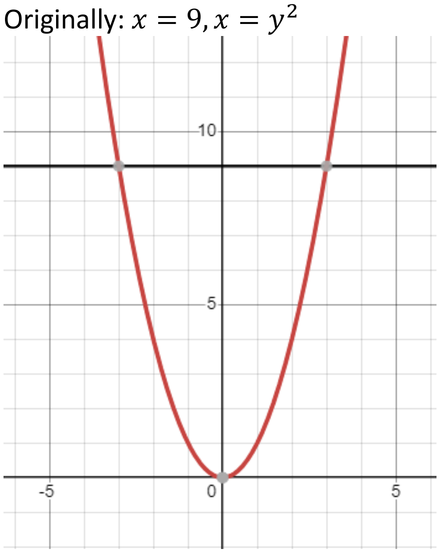
(second example)… the rightmost function is now the top function, and the leftmost function is now the bottom function.

Another option is to switch all your x-variables for y-variables, and all your y-variables for x-variables.

This previous example would look like:

$$y = 9, y = x^2$$

Originally: $x = 9, x = y^2$



If you do this correctly, the region will have the same area as the original.

Next time we'll talk about solids of revolution.