8/31/2023

Work
Probability Applications

Work: distance is constant and force is constant: $W = Fd$
Now, we are going to work variable force and variable distance.

First set: springs and gravity (variable force)
Second scenario: chains: variable distance, but also variable force (weight will depend on density)
Third scenario: tank (pumping water over the top of a tank), this will depend on the shape of the tank: variable distance, and variable force (function)

Springs: Hook's Law $F = kx$
(In some of the online textbook problems, they will give you a different formula for the spring force and you should use that instead of Hook's law).
Typically, the problem will include information on how to find the constant k, and then you will integrate the resulting force function.

$$W = \int_a^b kx\,dx$$

The think to keep in mind here: find k, check units, find the limits of integration.

Example. Suppose it takes a force of 10 N to compress (or stretch) a spring from equilibrium by 0.2 m. How much work is done compressing the spring 0.5 m from equilibrium?

$$10 = k\,(0.2)$$
$$k = 50$$

$$W = \int_0^{0.5} 50x\,dx = 50\left[\frac{1}{2}x^2\right]_0^{0.5} = 25(0.25) = \frac{25}{4} = 6.25\ Nm$$

$$g = 9.8\frac{m}{s^2}\ or\ 32\frac{ft}{s^2}$$

Example. Suppose you want to lift a 10-ton satellite into an orbit that is 1,100 miles above the surface of the Earth. (ignore the loss of propellent and any air resistance.) What is the work done to lift the satellite to the desired orbit. (Assume the radius of the Earth is approximately 4000 miles.)

Recall that the force of gravity formula is $F = \frac{Gm_1m_2}{d^2} = \frac{k}{x^2}$

Distance is the distance between the center of the two masses, so here, at the start, on the surface of the earth, the distance is the radius of the Earth: 4000 miles.

$$10 = \frac{k}{(4000)^2}$$

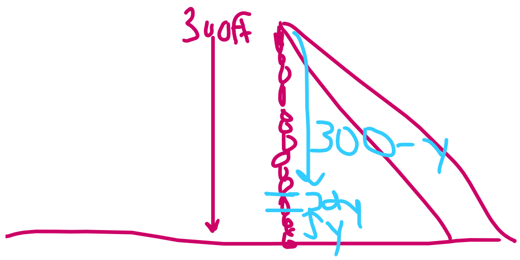
$$k = 1.6 \times 10^8$$

$$W = \int_{4000}^{5100} 1.6 \times 10^8\, x^{-2} dx = 1.6 \times 10^8 \left[-\frac{1}{x}\right]_{4000}^{5100} = 1.6 \times 10^8 \left[\frac{1}{4000} - \frac{1}{5100}\right]$$

$$= 8867.45\ ton - miles$$

Chains.

Example. Suppose a chain of length 300 feet hangs from a crane (the top of the crane is also 300 feet above the ground). The density of the chain is 5 lbs per foot. Find the force done in winding up the chain to the top of the crane.



How much does the little segment of the chain weigh? Weight=force=dy*density=$5dy$

How far is the segment being moved? $300 - y$

$$W = \int_0^{300} (300 - y)5dy = 5\left[300y - \frac{1}{2}y^2\right]_0^{300} = 5[90{,}000 - 45{,}000] = 225{,}000\ foot - pounds$$

If there is, say, a wrecking ball hanging from the end of the chain, so you can put this as a constant inside the integral or use the W=Fd formula just add the result to the rest of the chain problem.

Tank problems: pumping water out over the top of the tank.

Start with simple tank configurations: rectangular tanks or cylindrical tanks (the cross-sectional slices are all the same size).

Cylindrical tank Example.

Suppose that you have a cylindrical tank that is 10 meters across and 15 meters high. Find the work done pumping all of the water out of a full tank (over the top of the tank).

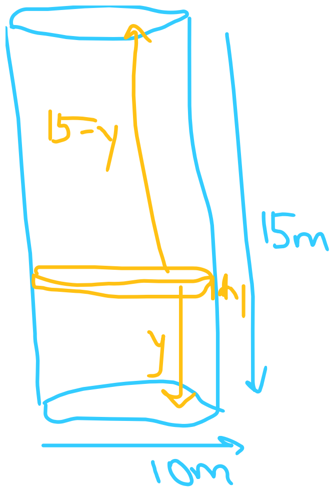
If the diameter is 10 m, then the radius is 5 m. And the volume of a cylinder is $V = \pi r^2 h$

$$V = \pi(5)^2 dy$$

Convert the volume to a weight: volume times the volume density of water.

In SI= 1000 kg/m^3 x 9.8 =9800

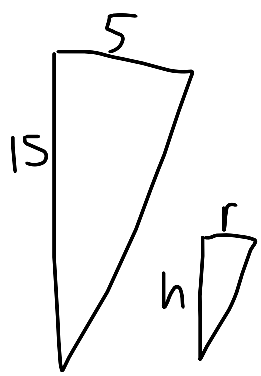
In English units: density of water in pounds per cubic foot = 62.4 lbs/ft^3

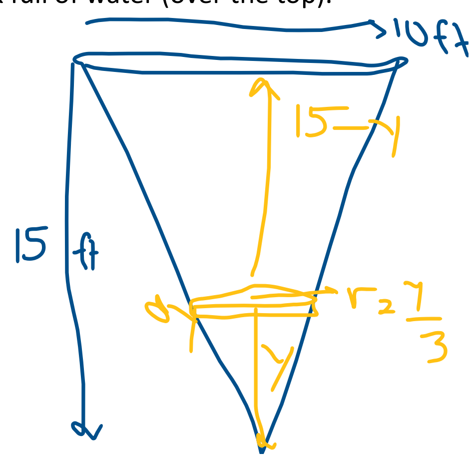


$$W = \int_0^{15} (15 - y) 9800 (25\pi) dy = 245{,}000\pi \int_0^{15} 15 - y \, dy = 245{,}000\pi \left[15y - \frac{1}{2} y^2 \right]_0^{15} =$$

$$245{,}000\pi \left[225 - \frac{225}{2} \right] = 245{,}000\pi \left(\frac{225}{2} \right) = 27{,}562{,}500\pi \, Nm$$

Conical tank:
Suppose I have a conical tank (point down) that is 15 feet deep and has a diameter of 10 feet at the top. Find the work done in pumping out the entire tank full of water (over the top).





$$\frac{15}{5} = \frac{h}{r} \to 3r = h = y \to r = \frac{y}{3}$$

$$V = \pi r^2 h = \pi \left(\frac{y}{3} \right)^2 dy$$

Force=weight=volume times density

$$F = \frac{\pi}{9} y^2 (62.4) dy$$

Distance to be moved: 15-y

$W = \int_0^{15} 62.4 \frac{\pi}{9} y^2 (15 - y) dy = \frac{62.4\pi}{9} \left[\frac{15}{3} y^3 - \frac{y^4}{4} \right]_0^{15} = \frac{62.4\pi}{9} \left[16875 - \frac{50625}{4} \right] = 29{,}250\pi$ foot-pounds

Probability Applications

In probability, for a continuous probability distribution, the area under the curve measures the probability, and the total area under the curve is required to be equal to 1 (since all probabilities must add to 1). A valid probability density function must have this property that the area where it is defined is exactly 1.

Consider the probability density function $f(x) = kx^2$ on the interval [0,3].

$$1 = \int_0^3 kx^2 dx = \frac{k}{3}x^3\Big|_0^3 = \frac{k}{3}(27) = 9k$$

$$9k = 1$$

$$k = \frac{1}{9}$$

$$f(x) = \frac{1}{9}x^2$$

Suppose we want to find the probability that x is between 1 and 2? $P(1 < X < 2)$?

$$P(1 < X < 2) = \int_1^2 \frac{1}{9}x^2 dx = \frac{1}{9}\left(\frac{1}{3}x^3\right)\Big|_1^2 = \frac{1}{27}[8-1] = \frac{7}{27}$$

$$P(X > 2) = \int_2^3 \frac{1}{9}x^2 dx$$

The cumulative distribution function: $F(x) = \int_0^x \frac{1}{9}t^2 dt = \frac{1}{27}x^3$

$$F(2) = P(X < 2)$$

To find the median of the distribution, the median is where the area is equal to ½ = 50% (the procedure is the same for any other percentile)

$$0.5 = \int_0^x \frac{1}{9}t^2 dt$$

$$\frac{1}{27}x^3 = 0.5$$

$$x^3 = 13.5$$

$$x = \sqrt[3]{13.5} \approx 2.3811 \ldots$$

After we talk about centers of mass on Tuesday, we'll talk about how to calculate the mean and the variance.

$$\mu = E(X) = mean = \int_a^b xf(x)dx = \int_0^3 \frac{1}{9}x^3 dx$$

$$\sigma^2 = variance = \int_a^b (x-\mu)^2 f(x)dx = \int_0^3 (x-\mu)^2 \left(\frac{1}{9}x^2\right) dx$$

$$= E(X^2) - [E(X)]^2 = \int_a^b x^2 f(x)dx - \mu^2 = \int_0^3 \frac{1}{9}x^4 dx - \mu^2$$

There is a relationship between the center of mass and the mean of a probability distribution.