9/12/2023

Trigonometric Substitution

Review trig integration:

A. Sine and cosine
   1) If sine is odd, pull out one sine and then convert the remaining sines to cosine
   2) If cosine is odd, pull out one cosine, and then convert the remaining cosines to sines
   3) If they are both even, then use the power-reducing identity until all terms are linear
B. Tangent and secant
   1) Secant is even, pull out two secants, then convert the remaining secants to tangents
   2) If tangent is odd, pull out one secant and one tangent, and then convert the remaining tangents to secants
   3) If secant is odd (with or without even tangents) then use integration by parts

Cosecant and Cotangent behave like secant and tangent.
If there are three different trig functions or a combination of trig functions that do not share a Pythagorean identity, then convert everything to sine and cosine.

Trig Substitution

The idea behind trig substitution is that we have radicals (with squares under the radicals) that we can't integrate by any other means. An example might be $\int \frac{x^2}{\sqrt{x^2+4}} dx$.

Look first to see if the integral can be done with a basic inverse trig rule first.

The form of the radical determines what substitution we choose:

$$\sqrt{a^2 - x^2} \rightarrow x = a \sin\theta$$

$$\sqrt{a^2 + x^2} \rightarrow x = a \tan\theta$$

$$\sqrt{x^2 - a^2} \rightarrow x = a \sec\theta$$

Example.

$$\int \sqrt{4 - x^2} dx$$

$$a = 2, x = 2\sin\theta\,, dx = 2\cos\theta\, d\theta,$$
$$\sqrt{4 - x^2} = \sqrt{4 - 4\sin^2\theta} = \sqrt{4(1 - \sin^2\theta)} = \sqrt{4\cos^2\theta} = 2\cos\theta$$

$$\int 2\cos\theta\,(2\cos\theta\, d\theta) = \int 4\cos^2\theta\, d\theta$$

$$\int 4\left(\frac{1}{2}\right)(1 + \cos 2\theta) d\theta = 2\int 1 + \cos 2\theta\, d\theta = 2\left[\theta + \frac{1}{2}\sin 2\theta\right] + C = 2\theta + \sin 2\theta + C$$

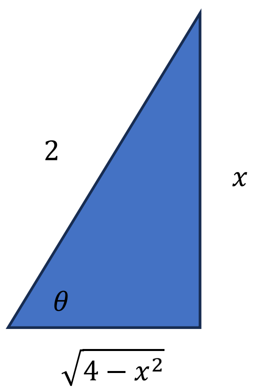
$$x = 2\sin\theta$$
$$\frac{x}{2} = \sin\theta$$

$$\arcsin\left(\frac{x}{2}\right) = \theta$$

Recall: $\sin 2\theta = 2\sin\theta\cos\theta$

$$\sqrt{4-x^2} = 2\cos\theta$$

$$\cos\theta = \frac{\sqrt{4-x^2}}{2}$$



$$\int \sqrt{4-x^2}\,dx = 2\theta + \sin 2\theta + C = 2\arcsin\left(\frac{x}{2}\right) + \frac{x\sqrt{4-x^2}}{2} + C$$

Example.

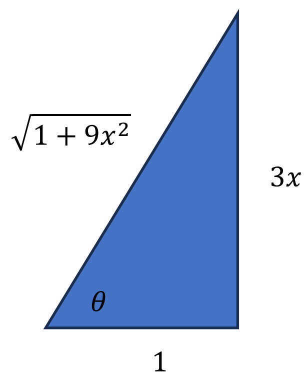
$$\int \frac{1}{\sqrt{1+9x^2}}\,dx$$

$$a = 1, u = 3x, 3x = \tan\theta\,, 3dx = \sec^2\theta\,d\theta, dx = \frac{1}{3}\sec^2\theta\,d\theta$$
$$\sqrt{1+9x^2} = \sqrt{1+(3x)^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = \sec\theta$$

$$\int \frac{\frac{1}{3}\sec^2\theta\,d\theta}{\sec\theta} = \int \frac{1}{3}\sec\theta\,d\theta = \frac{1}{3}\ln|\sec\theta + \tan\theta| + C =$$

$$\frac{1}{3}\ln\left|\sqrt{1+9x^2} + 3x\right| + C$$

$$\frac{3x}{1} = \tan\theta$$



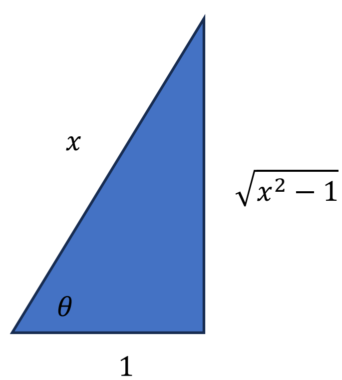
Example.

$$\int \frac{\sqrt{x^2-1}}{x^2}\,dx$$

$$a=1, x=\sec\theta\,, dx=\sec\theta\tan\theta\,d\theta$$
$$\sqrt{x^2-1}=\sqrt{\sec^2\theta-1}=\sqrt{\tan^2\theta}=\tan\theta$$

$$\int \frac{\tan\theta\sec\theta\tan\theta\,d\theta}{\sec^2\theta}=\int\frac{\tan^2\theta}{\sec\theta}d\theta=\int\frac{\sec^2\theta-1}{\sec\theta}d\theta=\int\frac{\sec^2\theta}{\sec\theta}-\frac{1}{\sec\theta}d\theta=$$

$$\int \sec\theta-\cos\theta\,d\theta=\ln|\sec\theta+\tan\theta|-\sin\theta+C$$
$$\frac{x}{1}=\sec\theta$$



$$\int\frac{\sqrt{x^2-1}}{x^2}dx=\ln|\sec\theta+\tan\theta|-\sin\theta+C=$$

$$\ln\left|x+\sqrt{x^2-1}\right|-\frac{\sqrt{x^2-1}}{x}+C$$

Example.

$$\int\sqrt{1+x^2}dx$$

$$a = 1, x = \tan\theta\,, dx = \sec^2\theta\, d\theta, \sqrt{1+x^2} = \sec\theta$$

$$\int \sec\theta \sec^2\theta\, d\theta = \int \sec^3\theta\, d\theta$$

$$u = \sec\theta\,, dv = \sec^2\theta\, d\theta$$
$$du = \sec\theta\tan\theta\, d\theta, v = \tan\theta$$

$$\sec\theta\tan\theta - \int \tan\theta\sec\theta\tan\theta\, d\theta = \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\, d\theta =$$

$$\sec\theta\tan\theta - \int (\sec^2\theta - 1)\sec\theta\, d\theta = \sec\theta\tan\theta - \int \sec^3\theta - \sec\theta\, d\theta =$$

$$\sec\theta\tan\theta + \int \sec\theta\, d\theta - \int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| - \int \sec^3\theta\, d\theta$$

$$\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| - \int \sec^3\theta\, d\theta$$

Add the secant-cubed integral to both sides

$$2\int \sec^3\theta\, d\theta = \sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + C$$
$$\int \sec^3\theta\, d\theta = \frac{1}{2}[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|] + C$$

$$\int \sqrt{1+x^2}dx = \frac{1}{2}[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|] + C =$$
$$\frac{1}{2}\left[x\sqrt{1+x^2} + \ln\left|\sqrt{1+x^2} + x\right|\right] + C$$

Example.

$$\int x^3(9-x^2)^{\frac{3}{2}}dx$$

$$a = 3, x = 3\sin\theta\,, dx = 3\cos\theta\, d\theta, \sqrt{9-x^2} = 3\cos\theta$$

$$\int (3\sin\theta)^3\,(3\cos\theta)^3 3\cos\theta\, d\theta = 3^7\int \sin^3\theta\cos^4\theta\, d\theta = 3^7\int \sin\theta\,(\sin^2\theta)\cos^4\theta\, d\theta =$$

$$3^7\int (1-\cos^2\theta)\cos^4\theta\,(\sin\theta\, d\theta) = 3^7\int \cos^4\theta - \cos^6\theta\,(\sin\theta\, d\theta) =$$

$$u = \cos\theta\,, du = -\sin\theta\, d\theta$$

$$3^7\int u^6 - u^4 du = 3^7\left[\frac{1}{7}u^7 - \frac{1}{5}u^5\right] + C = 3^7\left[\frac{1}{7}\cos^7\theta - \frac{1}{5}\cos^5\theta\right] + C =$$

$$\frac{\sqrt{9-x^2}}{3} = \cos\theta$$

$$3^7\left[\frac{1}{7}\left(\frac{\sqrt{9-x^2}}{3}\right)^7 - \frac{1}{5}\left(\frac{\sqrt{9-x^2}}{3}\right)^5\right] + C = \frac{1}{7}(9-x^2)^{7/2} - \frac{9}{5}(9-x^2)^{5/2} + C$$

Example. With completing the square.

$$\int \frac{1}{\sqrt{x^2+4x-12}}dx$$

$$x^2+4x-12 = (x^2+4x+4) - 12 - 4 = (x+2)^2 - 16$$

$$\int \frac{1}{\sqrt{x^2+4x-12}}dx = \int \frac{1}{\sqrt{(x+2)^2-16}}dx$$

$$a = 4, u = x+2, x+2 = 4\sec\theta, \sqrt{(x+2)^2-16} = 4\tan\theta, dx = 4\sec\theta\tan\theta\, d\theta$$

$$\int \frac{4\sec\theta\tan\theta\, d\theta}{4\tan\theta} = \int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C =$$

$$\ln\left|\frac{x+2}{4} + \frac{\sqrt{x^2+4x-12}}{4}\right| + C$$

Try to see if you can do the problem with a simpler method before using trig sub.

$$\int \frac{x^2}{x^2+4}dx$$

Don’t do trig sub here (you can, but don’t). Instead, do long division.

$$\int \frac{x}{x^2+4}dx$$

You could, but don’t. this is u-sub

$$\int x^2\sqrt{x^2-1}dx$$

You can do trig sub, but integration by parts, maybe, or change of variables

$$\int \frac{1}{x\sqrt{x^2-1}}dx$$

This is just arcsecant. Don’t both with trig sub.