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Integration Methods Overview
Integration by Tables
Numerical Integration

Strategies for Choosing the correction integration technique

In general, look to easier methods before more advanced methods:
Eliminate basic rules before any other methods
Does u-substitution work here?

$$f\big(g(x)\big)g'(x)$$
$$\frac{g'(x)}{g(x)}$$
$$g(x)g'(x)$$

Change of variable: you have an algebraic function containing a radical (with linear function on inside) and typically a polynomial – usually these can be done by integration by parts, so this is somewhat optional.

Integration by parts – rule of thumb LIATE (log-inverse trig-algebraic-trig-exponential)

Trig substitution – radicals with squares underneath.

Partial fractions – these are rational functions (first check to simplify, and the denominator has to factor)

Go back and consider is there an algebraic approach to simplifying the problem that could help here?

If all these fail, then we strategies left that might apply:
1) Table of integrals
2) If it's a definite integral (limits) then we can estimate the area under the curve numerically.

Examples.

$$\int xe^{x}dx \ \ vs. \int xe^{x^2}dx \ \ vs. \int x^2e^{x^2}dx$$

First: integration by parts, second: u-sub, third: can't be done.

$$\int x\sqrt{x^2+1}dx \ \ vs. \ \int x^2\sqrt{x^2+1}dx \ \ vs. \int \sqrt{x^2+1}\,dx \ \ \ vs. \int x^2\sqrt{x+1}dx$$

First: u-sub, second: trig-sub, third: trig-sub, fourth: could be integration by parts, or change of variables

Example.

$$\int \frac{xe^{2x}}{(2x+1)^2}dx$$

There are three functions: $x, e^{2x}, \frac{1}{(2x+1)^2}$

$$u = x, dv = \frac{e^{2x}}{(2x+1)^2}???$$
$$u = e^{2x}, dv = \frac{x}{(2x+1)^2}???$$
$$u = \frac{1}{(2x+1)^2}, dv = xe^{2x}????$$

dv is complicated and can't be integrated easily.

$$u = xe^{2x}, dv = \frac{1}{(2x+1)^2}???$$
$$u = \frac{x}{(2x+1)^2}, dv = e^{2x}???$$
$$u = \frac{e^{2x}}{(2x+1)^2}, dv = x???$$

Integration by Tables

Example.

$$\int \frac{\sqrt{4-9x^6}}{x^4}dx = \int \frac{x^2\sqrt{4-9x^6}}{x^2x^4}dx = \int \frac{x^2\sqrt{4-9x^6}}{x^6}dx = \int \frac{\frac{1}{9}\sqrt{2^2-u^2}}{\frac{1}{9}u^2}du = \int \frac{\sqrt{2^2-u^2}}{u^2}du$$

Let $u = 3x^3$ $(u^2 = 9x^6), 9x^2dx = du \rightarrow x^2dx = \frac{1}{9}du, u^2 = 9x^6 \rightarrow \frac{1}{9}u^2 = x^6$

(3.31) $$\int \frac{\sqrt{a^2 \pm x^2}}{x}dx = \sqrt{a^2 \pm x^2} - a\ln\left|\frac{a+\sqrt{a^2 \pm x^2}}{x}\right| + C, a > 0$$

(3.32) $$\int \frac{\sqrt{x^2 - a^2}}{x}dx = \sqrt{x^2 - a^2} - a\,\mathrm{arcsec}\left|\frac{x}{a}\right| + C$$

(3.33) $$\int \frac{\sqrt{x^2 \pm a^2}}{x^2}dx = -\frac{\sqrt{x^2 \pm a^2}}{x} + \ln\left|x + \sqrt{x^2 \pm a^2}\right| + C$$

(3.34) $$\int \frac{\sqrt{a^2 - x^2}}{x^2}dx = -\frac{\sqrt{a^2 - x^2}}{x} - \arcsin\left(\frac{x}{a}\right) + C$$

$$a = 2$$
$$-\frac{\sqrt{2^2-u^2}}{u} - \arcsin\left(\frac{u}{2}\right) + C = -\frac{\sqrt{4-9x^6}}{3x^3} - \arcsin\left(\frac{3x^3}{2}\right) + C$$

Example.

$$\int \tan^5 3x\, dx$$

(4.36) $$\int \tan^n x dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x dx, n \neq 1$$

$$u = 3x, du = 3dx \to \frac{1}{3}du = dx, n = 5$$

$$\frac{1}{3}\int \tan^5 u\, du = \frac{1}{3}\left[\frac{1}{4}\tan^4 u - \int \tan^3 u\, du\right] = \frac{1}{12}\tan^4 3x - \frac{1}{3}\int \tan^3 3x\, dx$$

(4.35) $$\int \tan^3(ax)dx = \frac{1}{a}\ln|\cos(ax)| + \frac{1}{2a}\sec^2(ax) + C$$

$$a = 3$$

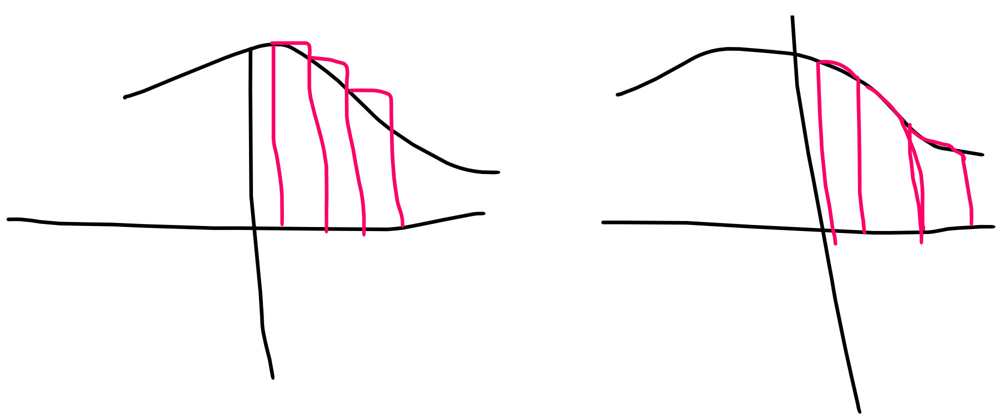
$$\frac{1}{12}\tan^4 3x - \frac{1}{3}\int \tan^3 3x\, dx = \frac{1}{12}\tan^4 3x - \frac{1}{3}\left[\frac{1}{3}\ln|\cos 3x| + \frac{1}{6}\sec^2 3x\right] + C$$

Numerical Integration

Is for estimating the area under a curve with greater precision (and fewer steps) than regular Reimann sums (apply to definite integrals)

Trapezoidal Rule, Simpson's Rule

Instead of using rectangles to approximate the area under the curve, we use a series of trapezoids.



$$A = \frac{1}{2}(b_1 + b_2)h$$

$$\int_a^b f(x)dx \approx \frac{b-a}{2n}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + f(x_n)]$$

Simpson's Rule:
Is approximating the curve with a quadratic function

$$\int_a^b f(x)dx \approx$$

$$\frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Carry decimals places until the end.
For Simpson's rule, n must be even

Approximate the area under the curve $f(x) = \ln x$, on the interval [1,2], using 4 rectangles.

$$b = 2, a = 1, n = 4$$

Trapezoidal rule:

$$\int_1^2 \ln x\, dx = \frac{2-1}{2(4)}[f(1) + 2f(1.25) + 2f(1.5) + 2f(1.75) + f(2)] =$$

$$\frac{1}{8}[\ln 1 + 2\ln 1.25 + 2\ln 1.5 + 2\ln 1.75 + \ln 2] \approx 0.383699 \ldots$$

Simpson's Rule:

$$\int_1^2 \ln x\, dx = \frac{2-1}{3(4)}[f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + f(2)] =$$

$$\frac{1}{12}[\ln 1 + 4\ln 1.25 + 2\ln 1.5 + 4\ln 1.75 + \ln 2] \approx 0.3862595 \ldots$$

"True" value is about 0.3862943…

Next time we'll look at the errors for these estimates and how to determine what value of n to use for an estimate with a specific level of accuracy.